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THERMAL STRAIN ANALYSIS OF ADVANCED MANNED SPACECRAFT HEAT SHIELDS

Final Report

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ABSTRACT

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Numerical methods and computer programs are presented for the analysis of heat shields. The finite element technique is used to determine stresses and displacements developed in composite axisymmetric solids of arbitrary geometry subjected to axisymmetric thermal or mechanical loads. This technique is then applied to the development of an automated computer program for the analysis of axisymmetric heat shields subjected to axisymmetric thermal and pressure loadings. Finally, the numerical technique is extended to the analysis of heat shields subjected to non-axisymmetric thermal loading.

Several examples are presented to illustrate the application of the method and to demonstrate its validity. FORTRAN II card listings and descriptions of the use of the above programs are given in the appendices.

Author

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LIST OF SYMBOLS

a_j, b_j, a_k, b_k = Element Dimensions

E = Modulus of Elasticity

u = Displacement in the r Direction

v = Displacement in the θ Direction

w = Displacement in the z Direction

α = Thermal Coefficient of Expansion

β = Over-Relaxation Factor

γ = Shear Strain

$\epsilon_{r,\theta,z}$ = Strain in the Radial, Circumferential and Longitudinal Direction

ν = Poisson's Ratio

$\sigma_{r,\theta,z}$ = Stress in the Radial, Circumferential and Longitudinal Direction

τ = Shearing Stress

$[\ ]^T$ = Matrix Transpose

$[a]$ = Displacement Transformation Matrix

$[C]$ = Matrix of Elastic Coefficients

$[G_n]$ = Displacement Transformation Matrix - Harmonic n

$[S]$ = Matrix of Element Corner Forces

$[u]$ = Matrix of Element Corner Displacements

$[k]$ = Element Stiffness Matrix

$[r]$ = Nodal Point Displacements

$[R]$ = Nodal Point Loads

$[K]$ = Stiffness Matrix for Complete Structure

INTRODUCTION

The purpose of this investigation is the development of methods of analysis and digital computer programs to aid in establishing the structural integrity of manned spacecraft heat shields. The results of the analysis which are presented in this report indicate only the capabilities of the computer programs and do not necessarily represent the behavior of a specific heat shield. The final evaluation of the structural capability of a heat shield must be based on a certain amount of engineering judgement, in connection with the use of the computer programs.

In this investigation the finite element method is used to determine stresses and displacements developed in solids of revolution. First, a numerical procedure and a digital computer program are developed for the analysis of composite axisymmetric solids of arbitrary geometry subjected to axisymmetric thermal or mechanical loads. Second, this program is specialized to the analysis of axisymmetric heat shields. Finally, the same numerical technique is extended to the analysis of heat shields subjected to non-axisymmetric thermal loading. A description of the method of analysis and the use of the above computer programs is presented. In addition, FORTRAN II listings of the above programs are incorporated in this report.

During the initial phases of this contract, finite difference techniques were used to solve the governing differential equations for displacements of the system. However, considerable difficulty was encountered in the solution of the resulting set of linear equations. An iterative approach, coupled with over-relaxation techniques, resulted in inadequately convergent displacements. The direct solution technique gave a matrix for the set of simultaneous equations which was ill-conditioned. An additional difficulty of the finite difference technique was encountered in satisfying the boundary conditions at the edge of the heat shield. The finite element approach proved more practical and more versatile; therefore, the finite difference method was discontinued. A complete description of this initial investigation is given in Appendix G.

PART I: METHOD OF ANALYSIS

A. INTRODUCTION

The "finite element method" is a general method of structural analysis in which a continuous structure is replaced by a finite number of elements interconnected at a finite number of nodal points -- (such an idealization is inherent in the conventional analysis of frames and trusses). In this investigation the finite element method is applied to the determination of stresses and displacements developed in axisymmetric elastic structures of arbitrary geometry and material properties which are subjected to thermal and mechanical loads.

An assemblage of different types of axisymmetric elements is used to represent the continuous structure. Approximations are made on the displacements within each element of the system. Based on these approximations, equilibrium equations are developed for all elements. From "direct stiffness techniques", the equilibrium equations, in terms of unknown nodal point displacements, are developed at each nodal point. A solution of this set of equations constitutes a solution to the finite element system.

B. EQUILIBRIUM EQUATIONS FOR AN ARBITRARY FINITE ELEMENT

1. Strain-Displacement Relationship

The first step in the determination of the stiffness (corner forces in terms of corner displacements and temperature changes), of a finite element is to assume a solution for the displacement field within the element. It is desirable that this assumed displacement field satisfies compatibility between other elements in the system. Based on this solution for the displacements within the element, it is possible to develop an expression for the strains at any point within the element in terms of the nodal points (corner) displacements. This expression in matrix form is

$$\left[\epsilon\right] = \left[a\right]\left[u\right] \tag{1.1}$$

where $\left[\epsilon\right]$ is a column matrix of the M components of strain

$\left[u\right]$ is a column matrix of the N nodal point displacements

$\left[a\right]$ is an M x N strain-displacement transformation matrix - this matrix may be a function of space

2. <u>Stress-Strain Relationship</u>

For an elastic material, the stresses at any point within the element are expressed in terms of the corresponding strains by the elastic stress-strain relationship. Or, in matrix form

$$\left[\sigma\right] = \left[C\right]\left[\epsilon\right] + \left[\tau\right] \tag{1.2}$$

where $\left[\sigma\right]$ is a column matrix of the M components of stress

$\left[\epsilon\right]$ is a column matrix of the M components of strain

$\left[\tau\right]$ is a column matrix of the M components of thermal stress

$\left[C\right]$ is an MxM matrix of material property coefficients

The size (M) of these matrices will depend on the type of element being considered. The coefficients of matrices $\left[C\right]$ and $\left[\tau\right]$ will depend on material properties. Since $\left[C\right]$ is completely arbitrary, anisotropic materials can be handled. Also, each element in the system may have different properties; therefore, composite structures are readily represented by the finite element idealization.

3. Internal Work

The internal work, or strain energy, which is associated with an infinitesimal volume element dV within the finite element is given by

$$dW_I = \frac{1}{2} (\epsilon_1 \sigma_1 + \epsilon_2 \sigma_2 \cdots\cdots + \epsilon_M \sigma_M) \, dV$$

or in matrix form

$$dW_I = \frac{1}{2} \left[\epsilon\right]^T \left[\sigma\right] dV \tag{1.3}$$

The substitution of Equation (1.2) into Equation (1.3) yields

$$dW_I = \frac{1}{2} \left[\epsilon\right]^T \left[C\right] \left[\epsilon\right] dV + \frac{1}{2} \left[\epsilon\right]^T \left[\tau\right] dV \tag{1.4}$$

Equation (1.1) may be written in transposed form as

$$[\epsilon]^T = [u]^T [a]^T \tag{1.5}$$

After Equations (1.1) and (1.5) are substituted into Equation (1.4), the internal work is given by

$$dW_I = \frac{1}{2} [u]^T [a]^T [C] [a] [u] \, dV + \frac{1}{2} [u]^T [a]^T [\tau] \, dV \tag{1.6}$$

The total strain energy stored within the element is found by integrating Equation (1.6) over the volume of the finite element. Or

$$W_I = \frac{1}{2} [u]^T \int [a]^T [C] [a] [u] \, dV + \frac{1}{2} [u]^T \int [a]^T [\tau] \, dV \tag{1.7}$$

4. <u>External Work</u>

The work supplied externally at the nodal points of the finite element is given by

$$W_E = \frac{1}{2} U_1 S_1 + \frac{1}{2} U_2 S_2 \ldots\ldots\ldots\ldots + \frac{1}{2} U_N S_N$$

or in matrix form

$$W_E = \frac{1}{2} [u]^T [S] \tag{1.8}$$

where $[u]^T$ is a row matrix of the N nodal point displacements

$[S]$ is a column matrix of the N corresponding nodal point forces

5. Energy Balance

The external work, Equation (1.8), is equated to the internal work, Equation (1.7), yielding

$$\left[u\right]^T\left[S\right] = \left[u\right]^T\left[k\right]\left[u\right] + \left[u\right]^T\left[L\right] \tag{1.9}$$

where the element stiffness matrix

$$\left[k\right] = \int \left[a\right]^T\left[C\right]\left[a\right] dV \tag{1.10}$$

and the thermal load matrix

$$\left[L\right] = \int \left[a\right]^T\left[\tau\right] dV \tag{1.11}$$

Equation (1.9) represents an energy balance (scalar equation) for a single nodal point displacement pattern. If the final displacements $\left[u_i\right]$ are assumed to be composed of N separate displacement patterns, $\left[\bar{u}_{ij}\right]$ j = 1,...N, and if the final forces $\left[S_i\right]$ are assumed to be composed of N corresponding sets of forces, $\left[S_{ij}\right]$ j = 1, ... N, Equation (1.9) may be written as

$$\left[\bar{u}\right]^T\left[\bar{S}\right] = \left[\bar{u}\right]^T\left[k\right]\left[\bar{u}\right] + \left[\bar{u}\right]^T\left[L\right] \tag{1.12}$$

To eliminate the term $\left[\bar{u}\right]^T$, the displacement patterns must be selected in such a manner as to assure an inverse of $\left[\bar{u}\right]^T$. An acceptable matrix is a diagonal matrix of the final displacement, or

$$\left[\bar{u}\right]^T = \left[\bar{u}\right] = \left[\diagdown u \diagdown\right] \tag{1.13}$$

Equation (1.12) is now premultiplied by $\left[\diagdown u \diagdown\right]^{-1}$ yielding

$$\left[I\right]\left[\bar{S}\right] = \left[I\right]\left[k\right]\left[\diagdown u \diagdown\right] + \left[I\right]\left[L\right] \qquad (1.14)$$

where $\left[I\right]$ is a diagonal unit matrix.

Since only linear systems are considered, the N displacement patterns may be superimposed. Or

$$\left[S\right] = \left[k\right]\left[u\right] + \left[L\right] \qquad (1.15)$$

Since

$$S_i = \sum_{j=1,\ldots N} \bar{\bar{S}}_{ij} \quad \text{and} \quad u_i = \bar{\bar{u}}_{ii}$$

Equation (1.15) expresses nodal point forces in terms of nodal point displacements and temperature changes within the element.

C. EQUILIBRIUM EQUATIONS FOR A SYSTEM OF FINITE ELEMENTS

The first step in the procedure is to express <u>all</u> element forces in terms of external nodal point displacements for each element in the system. This is accomplished by expanding Equation (1.14) in terms of the N possible nodal point displacements; this will yield M matrix equations of the form

$$\left[S^m\right] = \left[k^m\right]\left[r\right] + \left[L^m\right] \qquad m = 1, \ldots. M \qquad (1.16)$$

where M is the total number of elements in the system.

The matrix $\left[k^m\right]$ is termed the <u>complete</u> stiffness of element m and involves only terms which are associated with the displacements of the connecting nodal points. Consequently, the majority of the coefficients of this matrix equation are zero. The matrix $\left[r\right]$ contains all possible nodal point displacements of the complete finite element system. The matrix $\left[S^m\right]$ is a column matrix containing the forces acting on element m in the direction of the nodal point displacements $\left[r\right]$. The thermal load matrix $\left[L^m\right]$ and the element stiffness matrix $\left[k^m\right]$ are given by Equations (1.11) and (1.10); however, the order (size) of these matrices has now been expanded to correspond with the total number of nodal point displacements.

In order to satisfy equilibrium of all nodal points, the sum of the internal element forces must be equal to the external nodal point loads. Or

$$\left[P\right] = \sum_{m=1,\ldots M} \left[S^m\right] \tag{1.17}$$

where $\left[P\right]$ is the externally applied nodal point loads. The substitution of Equation (1.16) into Equation (1.17) yields

$$\left[P\right] = \sum_{m=1,\ldots M} \left[k^m\right]\left[r\right] + \sum_{m=1,\ldots M} \left[L^m\right] \tag{1.18}$$

or rewritten in the following form:

$$\begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} K \end{bmatrix} \begin{bmatrix} r \end{bmatrix} \tag{1.19}$$

where

$$\begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} P \end{bmatrix} - \sum_{m=1,\ldots M} \begin{bmatrix} L^m \end{bmatrix} \tag{1.20}$$

$$\begin{bmatrix} K \end{bmatrix} = \sum_{m=1,\ldots M} \begin{bmatrix} k^m \end{bmatrix} \tag{1.21}$$

Equation (1.19), which is an equilibrium relationship between external loads and internal forces, represents a system of N linear equations in terms of N unknown displacements.

D. SOLUTION OF EQUILIBRIUM EQUATIONS

Equation (1.19) represents the relationship between all nodal point forces and all nodal point displacements. Mixed boundary conditions are considered by rewriting Equation (1.19) in the following partitioned form:

$$\left[\begin{array}{c} R_a \\ \hline R_b \end{array}\right] = \left[\begin{array}{c|c} K_{aa} & K_{ab} \\ \hline K_{ba} & K_{bb} \end{array}\right] \left[\begin{array}{c} r_a \\ \hline r_b \end{array}\right] \tag{1.22}$$

where $\left[R_a\right]$ = the specified nodal point forces

$\left[R_b\right]$ = the unknown nodal point forces

$\left[r_a\right]$ = the unknown nodal point displacements

$\left[r_b\right]$ = the specified nodal point displacements

Equation (1.22) may be expressed in terms of two separate equations, or

$$\left[R_a\right] = \left[K_{aa}\right]\left[r_a\right] + \left[K_{ab}\right]\left[r_b\right] \tag{1.23}$$

$$\left[R_b\right] = \left[K_{ba}\right]\left[r_a\right] + \left[K_{bb}\right]\left[r_b\right] \tag{1.24}$$

Equation (1.23) is rewritten in the following reduced form:

$$\left[K_{aa}\right]\left[r_a\right] = \left[\bar{\bar{R}}_a\right] \tag{1.25}$$

where the modified load vector, $\left[\bar{\bar{R}}_a\right]$ is given by

$$\left[\bar{\bar{R}}_a\right] = \left[R_a\right] - \left[K_{ab}\right]\left[r_b\right] \tag{1.26}$$

In Part II of this report, the Gauss-Seidel iterative technique is used to solve Equation (1.25) for the unknown nodal point displacements $\left[r_a\right]$. Appendix A gives a direct solution approach which is used in the automated computer programs for the thermal stress analysis of heat shields, Parts III and IV of this report.

E. ELEMENT STRESSES

After the nodal point displacements have been determined, the strains within any element in the system are evaluated by the direct application of Equation (1.1). The corresponding stresses are calculated from the stress-strain relationship, Equation (1.2).

PART II GENERAL COMPUTER PROGRAM FOR THE ANALYSIS OF ARBITRARY AXISYMMETRIC STRUCTURES

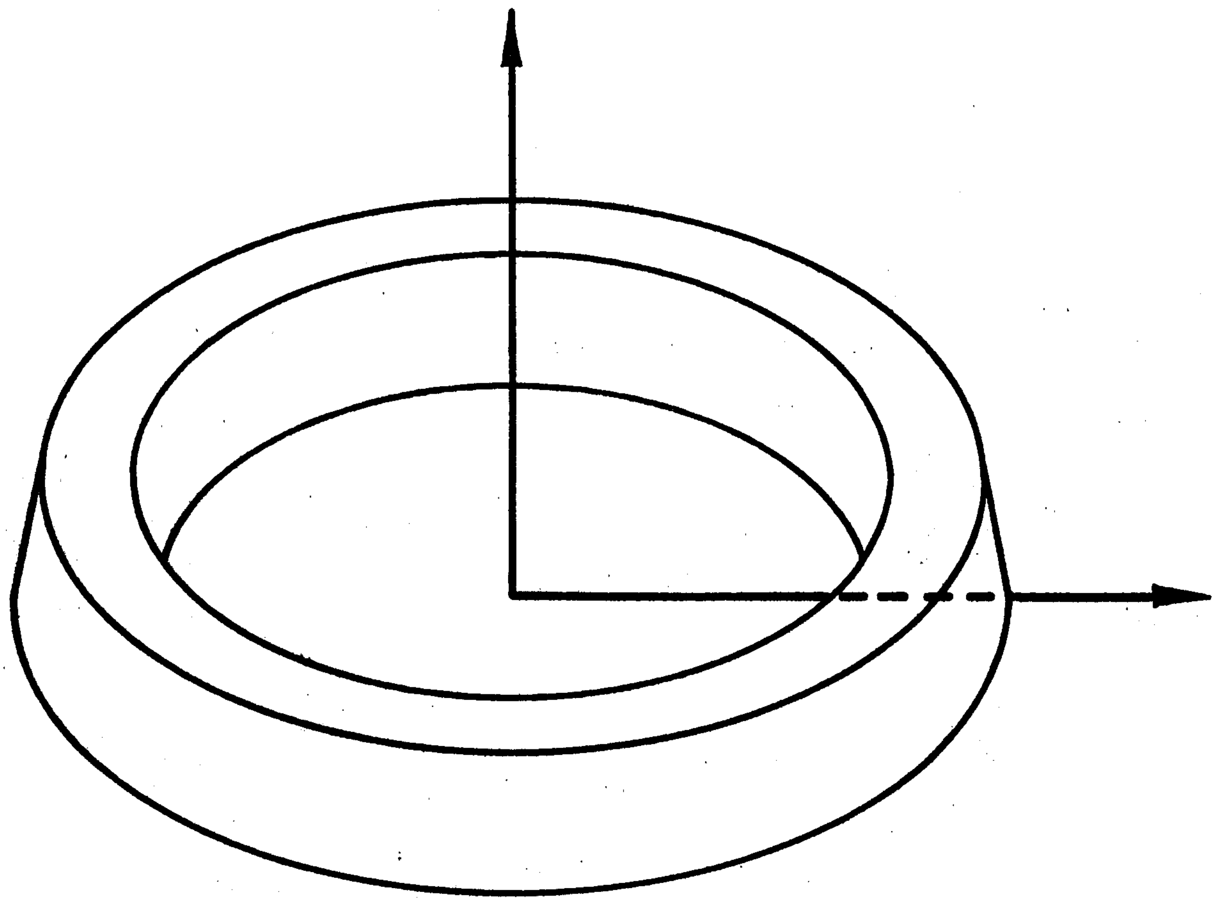
A. INTRODUCTION

The stress analysis of an axisymmetric structure of arbitrary shape, subjected to thermal and mechanical loads is of considerable practical interest. Although the governing differential equations have been known for many years, closed form solutions have been obtained for only a limited number of structures. Thus, the investigator must often rely on experimental or numerical procedures to solve this problem.

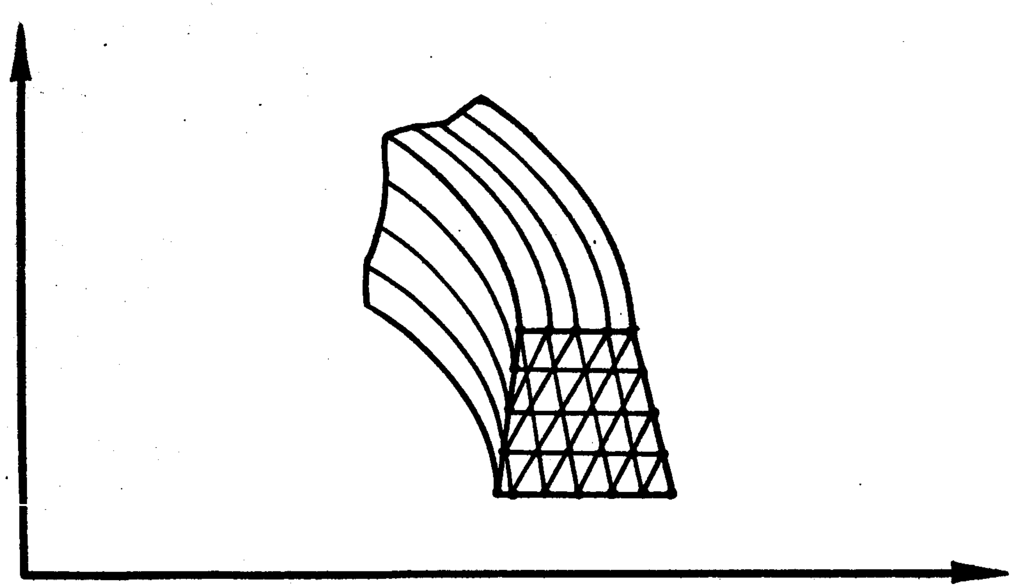
Experimental methods, such as Photoelasticity, have proven to be versatile tools in the analysis of many axisymmetric structures. However, for structures composed of several different materials or structures with thermal loading, this approach is limited.

The finite difference method, which involves the replacement of the derivatives in the differential equations and boundary conditions with difference equations, has been the most popular of the numerical techniques. However, for structures of composite materials and of arbitrary geometry, the procedure is difficult to apply.

In this section the finite element method is used to determine the stresses and displacements developed within arbitrary, elastic solids of revolution subjected to thermal or mechanical axisymmetric loads. The



a. ACTUAL RING



b. TRIANGULAR ELEMENT APPROXIMATION

FIG. 2.1 THE FINITE ELEMENT IDEALIZATION finite element approach replaces the continuous structure with a system of triangular rings interconnected at a finite number of nodal points (joints). Loads acting on the structure are replaced by statically equivalent concentrated forces acting at the nodal points of the finite element system. Figure 2.1 illustrates a finite element idealization of a typical axisymmetric solid.

B. STIFFNESS OF TRIANGULAR RING

1. Strain-Displacement Relationship

Continuity between elements of the system is maintained by requiring that within each element "lines initially straight remain straight in their displaced position". This linear displacement field, which is illustrated in Figure 2.2, is defined in terms of u(r,z) and

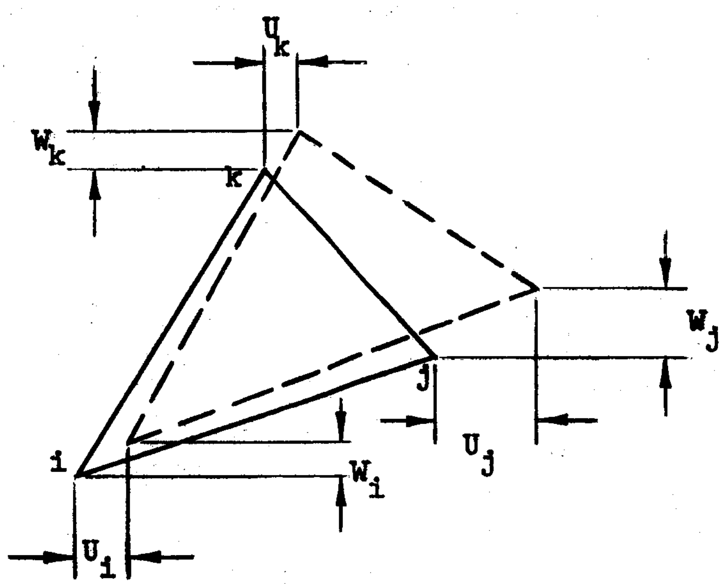


FIG. 2.2 ASSUMED DISPLACEMENT PATTERN

w(r,z) by equations of the following form:

$$u(r,z) = C_1 + C_2 r + C_3 z \qquad (2.1a)$$

$$w(r,z) = C_4 + C_5 r + C_6 z \qquad (2.1b)$$

If Equations (2.1a) and (2.1b) are evaluated at the three corners i, j, k of the triangle, the following set of equations is obtained:

$$\begin{bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_k \\ w_k \end{bmatrix} = \begin{bmatrix} 1 & r_i & z_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r_i & z_i \\ 1 & r_j & z_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r_j & z_j \\ 1 & r_k & z_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r_k & z_k \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} \qquad (2.2)$$

By solving the system of Equations (2.2) for the constants $C_1, \ldots, C_6$, they are expressed in terms of corner displacements. The strains in the rz-plane are obtained from the assumed displacement field by considering the basic definition of strain.

$$\bar{\epsilon}_r = \frac{\partial u}{\partial r} = C_2 \tag{2.3a}$$

$$\bar{\epsilon}_z = \frac{\partial w}{\partial z} = C_6 \tag{2.3b}$$

$$\bar{\gamma}_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = C_3 + C_5 \tag{2.3c}$$

At any point within the element the tangential strain $\bar{\epsilon}_\theta$ is

$$\bar{\epsilon}_\theta(r,z) = \frac{u(r,z)}{r}$$

The average tangential strain is found by averaging the strains at the vertices of the triangle, or

$$\epsilon_\theta = \frac{1}{3}\left(\frac{u_i}{r_i} + \frac{u_j}{r_j} + \frac{u_k}{r_k} \right) \tag{2.3d}$$

After eliminating the constants C_n between Equations (2.2) and (2.3), the average element strains are expressed in terms of corner displacements by the following matrix equation:

$$
\begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} b_j - b_k & 0 & b_k & 0 & -b_j & 0 \\ 0 & a_k - a_j & 0 & -a_k & 0 & a_j \\ \frac{\lambda}{3r_i} & 0 & \frac{\lambda}{3r_j} & 0 & \frac{\lambda}{3r_k} & 0 \\ a_k - a_j & b_j - b_k & -a_k & b_k & a_j & -b_j \end{bmatrix} \begin{bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_k \\ w_k \end{bmatrix} \qquad (2.4a)
$$

or in symbolic form

$$
\left[\epsilon\right] = \left[a\right]\left[u\right] \qquad (2.4b)
$$

where

$$
\begin{aligned}
a_j &= r_j - r_i \\
a_k &= r_k - r_i \\
b_j &= z_j - z_i \\
b_k &= z_k - z_i \\
\lambda &= a_j b_k - a_k b_j
\end{aligned}
$$

The geometry of a typical triangle is illustrated in Figure 2.3

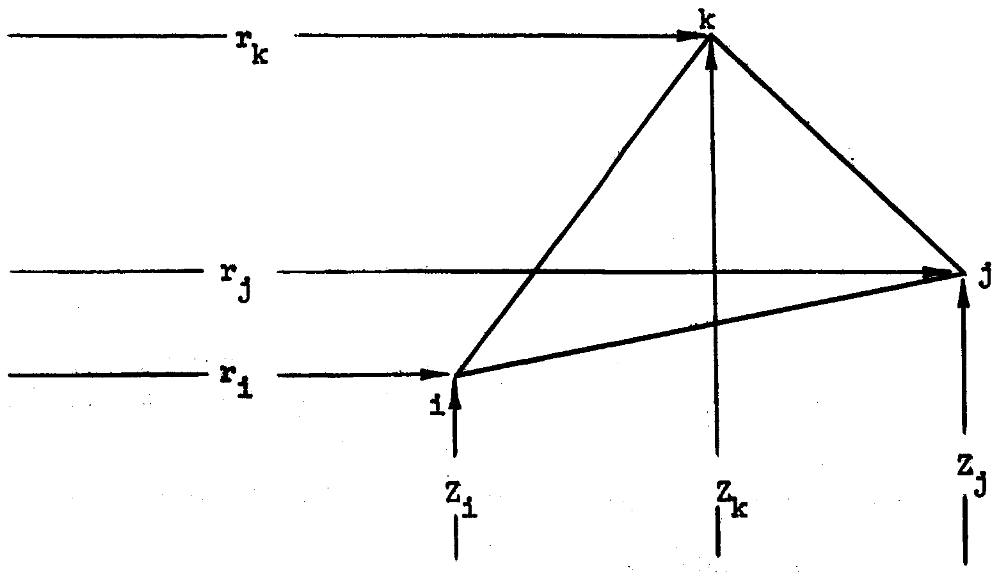


FIG. 2.3 ELEMENT DIMENSIONS

2. Stress-Strain Relationship

One important advantage of the finite element approach is that structures with anisotropic materials can be treated. In general, the stress-strain relationship is of the form

$$\begin{bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \sigma_{rz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & \epsilon_r \\ C_{21} & C_{22} & C_{23} & C_{24} & \epsilon_z \\ C_{31} & C_{32} & C_{33} & C_{34} & \epsilon_\theta \\ C_{41} & C_{42} & C_{43} & C_{44} & \gamma_{rz} \end{bmatrix} + \begin{bmatrix} \tau_r \\ \tau_z \\ \tau_\theta \\ \tau_{rz} \end{bmatrix} \tag{2.5a}$$

or symbolically $\left[\sigma\right] = \left[C\right]\left[\epsilon\right] + \left[\tau\right]$ (2.5b) where $\left[\tau\right]$ is the matrix of thermal stresses for a given temperature change. For example, the stress-strain relationship for an isotropic material is given by

$$\begin{bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \sigma_{rz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \end{bmatrix} + \begin{bmatrix} \tau \\ \tau \\ \tau \\ 0 \end{bmatrix} \qquad (2.6)$$

$$\text{where } \tau = \frac{E\alpha}{(1-2\nu)} \Delta T \qquad (2.7)$$

3. <u>Element Stiffness</u>

The stiffness of a typical triangular ring, which is an expression for corner forces in terms of corner displacements, is given by Eq. (1.10) as

$$\left[k\right] = \int \left[a\right]^T \left[c\right] \left[a\right] dV$$

And the thermal load matrix is given by Eq. (1.11) as

$$\left[L\right] = \int \left[a\right]^T \left[\tau\right] dV$$

Since the coefficients in matrices $\left[a\right]$ and $\left[C\right]$ are assumed not to be a function of space the above equations reduce to

$$[k] = V [a]^T [C] [a] \tag{2.8}$$

$$[L] = V [a]^T [\tau] \tag{2.9}$$

If a one-radian segment is considered, an approximate expression for the volume of a triangular ring segment is

$$V = \bar{r} A \tag{2.10}$$

where $\bar{r}$ is the average radius given by

$$\bar{r} = (r_i + r_j + r_k)/3 \tag{2.11}$$

and A is the cross-sectional area given by

$$A = (a_j b_k - a_k b_j)/2 \tag{2.12}$$

From Eq. (1.15) the six corner forces acting at the vertices of a one-radian triangular segment is given in terms of the six corner displacements and the temperature change within the element by

$$[S] = [k] [u] + [L] \tag{2.13}$$

C. EQUILIBRIUM EQUATIONS FOR COMPLETE STRUCTURE

The equilibrium of the complete system of triangular rings, which is an expression for nodal point loads in terms of nodal point displacements, is given by the following matrix equation:

$$[R] = [K] [r] \tag{2.14}$$

The stiffness matrix $\left[K\right]$ and the load matrix $\left[R\right]$ are determined by "direct stiffness" techniques as indicated in the previous section, Eqs. (1.20) and (1.21). In addition to the thermal loads, the $\left[R\right]$ matrix is composed of concentrated external forces acting at the nodal points of the system. Hence, pressures acting on the boundary of a segment of the structure are replaced by statically equivalent forces acting at the nodal points.

Mixed boundary conditions are considered by a simple transformation of Eq. (2.14); Eqs. (1.22) to (1.26) give the details of this modification.

D. DETERMINATION OF DISPLACEMENT AND STRESSES

Equation (2.14) is solved for the unknown nodal point displacements by the application of the well-known Gauss-Seidel iterative procedure. This involves the repeated calculation of new displacements from the equation

$$r_n^{(s)} = K_{nn}^{-1}\left[R_n - \sum_{i=1,..n-1} K_{ni} r_i^{(s)} - \sum_{i=n+1,..n} K_{ni} r_i^{(s-1)}\right. \qquad (2.15)$$

where n is the number of the unknown and s is the cycle of iteration.

The only modification of the procedure introduced in this analysis is the simultaneous application of Equation (2.15) to both components of displacements at each nodal point. Therefore, r_n and R_n become vectors with r and z components.

The rate of convergence of the Gauss-Seidel procedure can be greatly increased by the use of an over-relaxation factor. This factor is applied by first calculating the change in displacement $\Delta r_n^{(s)}$ of nodal point n and then determining the new displacement from the following equation:

$$r_n^{(s)} = r_n^{(s-1)} + \beta \Delta r_n^{(s)} \tag{2.16}$$

where β is the over-relaxation factor.

The solution of an over-relaxation factor, which gives the best convergence, depends on the characteristics of the particular problem. However, experience has indicated that for most structures, the optimum over-relaxation factor is between 1.8 and 1.95.

Since only the non-zero terms in Equation (2.14) are developed and stored by the computer program, a solution of several hundred equations is possible, thereby making it possible to solve large finite element systems.

For each element the average strains are calculated directly from the nodal point displacements by the application of Equation (2.4). The average element stresses are then determined from the stress-strain relationship for the element, Equation (2.5). In addition, at each nodal point, stresses are computed by averaging the stresses in all elements attached to the point.

E. COMPUTER PROGRAM

The complete analysis of an axisymmetric solid by the finite element method involves three separate phases. First, the structure must be idealized by a system of triangular rings. Second, this system is solved for displacements and stresses from given nodal point forces. Third, the displacements and stresses are presented graphically for further evaluation and utilization.

The selection of the system of finite elements for a particular problem is completely arbitrary; therefore, axisymmetric structures, composed of many interacting components, of practically any shape may be handled. By numbering all elements and nodal points, in a convenient manner, the system can be defined in the form of three numerical arrays - nodal point array, element array and boundary point array. The nodal point array contains the coordinates and the loads or displacements that are associated with each nodal point of the system. The element array contains, for each element in the system, the location of the element (the three nodal point numbers defining the corners of the element and other possible parameters which are associated with the element (i.e., elastic constants, density and temperature changes). The boundary array indicates the type of restraint that exists at boundary nodal points.

These three arrays, along with some basic control information, constitute the numerical input for the digital computer program. The program itself performs three major tasks in the analysis of the finite element system of triangular rings. First, the equilibrium equations for the system are formed from the basic numerical description of the system. Second, this set of equations is solved for the nodal point displacements. Third, the internal stresses are determined from these displacements.

1. Input Information

To define the system of finite elements, all nodal points and elements are numbered as illustrated in Figure 2.4. Based on this numbering system, the following sequence of punched cards constitutes the input to the computer program.

a. TITLE CARD (72H)

Columns 2 to 72 of this card contain information to be printed with results

b. CONTROL CARD (6I4, 2E12.5)

Columns	1 - 4	Number of elements
	5 - 8	Number of nodal points
	9 - 12	Number of restrained boundary points
	13 - 16	Cycle interval for print of unbalanced forces

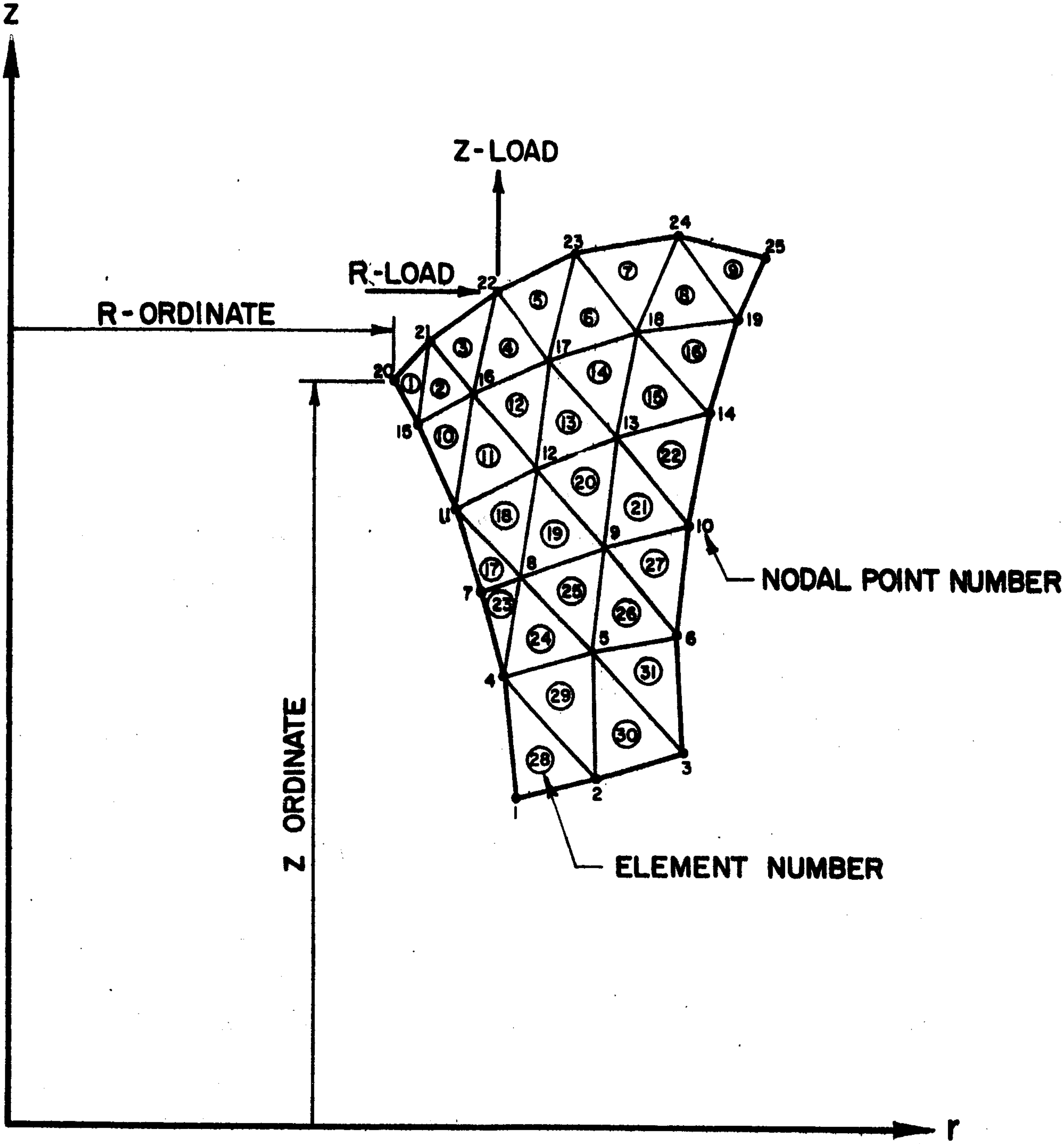


FIG. 2.4 NUMBERING SYSTEM FOR ELEMENTS AND NODAL POINTS

17-20 Cycle interval for print of results

21-24 Maximum number of cylces problem may run

25-36 Convergence limit for unbalanced forces

37-48 Over-relaxation factor

c. ELEMENT ARRAY - 1 card per element (4I4, 4E12.4, F8.4)

Column		
1-4	Element Number	
5-8	Nodal point number i	in counter-clockwise order
9-12	Nodal point number j	
13-16	Nodal point number k	
17-28	Modulus of elasticity E	
29-40	Density of element ρ	
41-52	Poisson's ratio ν	
53-64	Coefficient of thermal expansion α	
65-72	Temperature change within element ΔT	

d. NODAL POINT ARRAY - 1 card per nodal point (1I4, 4F8.1, 2F12.8)

Column		
1-4	Nodal point number	
5-12	R-ordinate	
13-20	Z-ordinate	
21-38	R-load	Total force acting on a one radian segment.
29-36	Z-load	
37-48	R-displacement	
49-60	Z-displacement	

On free nodal points, the displacements are initial guesses for the iterative solution. On restrained nodal points, the input displacements are the specified final displacements of the nodal point.

e. BOUNDARY POINT ARRAY - 1 card per restrained nodal point (2I4, IF6)

Columns	1-4	Nodal point numbers
	5-8	0 if point is fixed in both directions
		1 if point is fixed in the R-direction
		2 if point is free to move along a line of slope S
	9-16	Slope S (type 2 points only)

2. Output Information

The following information is generated and printed by the computer program:

a. Input Data
b. Nodal Point Displacement
c. Average Element Stresses
d. Average Nodal Point Stresses

3. Timing

For the IBM 7094 the computational time required by the program is approximately 0.004 x n x m seconds, where n equals the number of nodal points and m equals the number of cycles of iteration. Depending on the desired degree of convergence, it may be necessary to extend the iteration process.

4. Program Listing

A card listing of the FORTRAN II source deck for the general axisymmetric program is included in Appendix D of this report. This program is compiled for a maximum size of 550 elements or 340 nodal points.

F. EXAMPLE

An infinite cylinder subjected to steady state temperature distribution, for which an exact solution is known, is selected as a means of verifying the finite element analysis. A finite element idealization of a section of the cylinder is shown in Figure 2.5a. The temperature distribution which is assumed constant within each element, is plotted in Figure 2.5b. The hoop stresses are compared with the exact solution in Figure 2.5c. Considering the coarse mesh, agreement with the exact solution is very good except at the two boundary points. This discrepancy is due to the fact that nodal point stresses are calculated by averaging the stresses in the attached elements. Therefore, the boundary nodal point stress reflects the average stress in the elements near the boundary.

In general, good boundary stresses are obtained by plotting the interior stresses and extrapolating to the boundary. This type of engineering judgement is always necessary in evaluating results from a finite element analysis.

G. DISCUSSION

This section demonstrates the application of the finite element technique to the stress analysis of structures of revolution. The approach reduces the stress analysis to a simple procedure. In order to use the

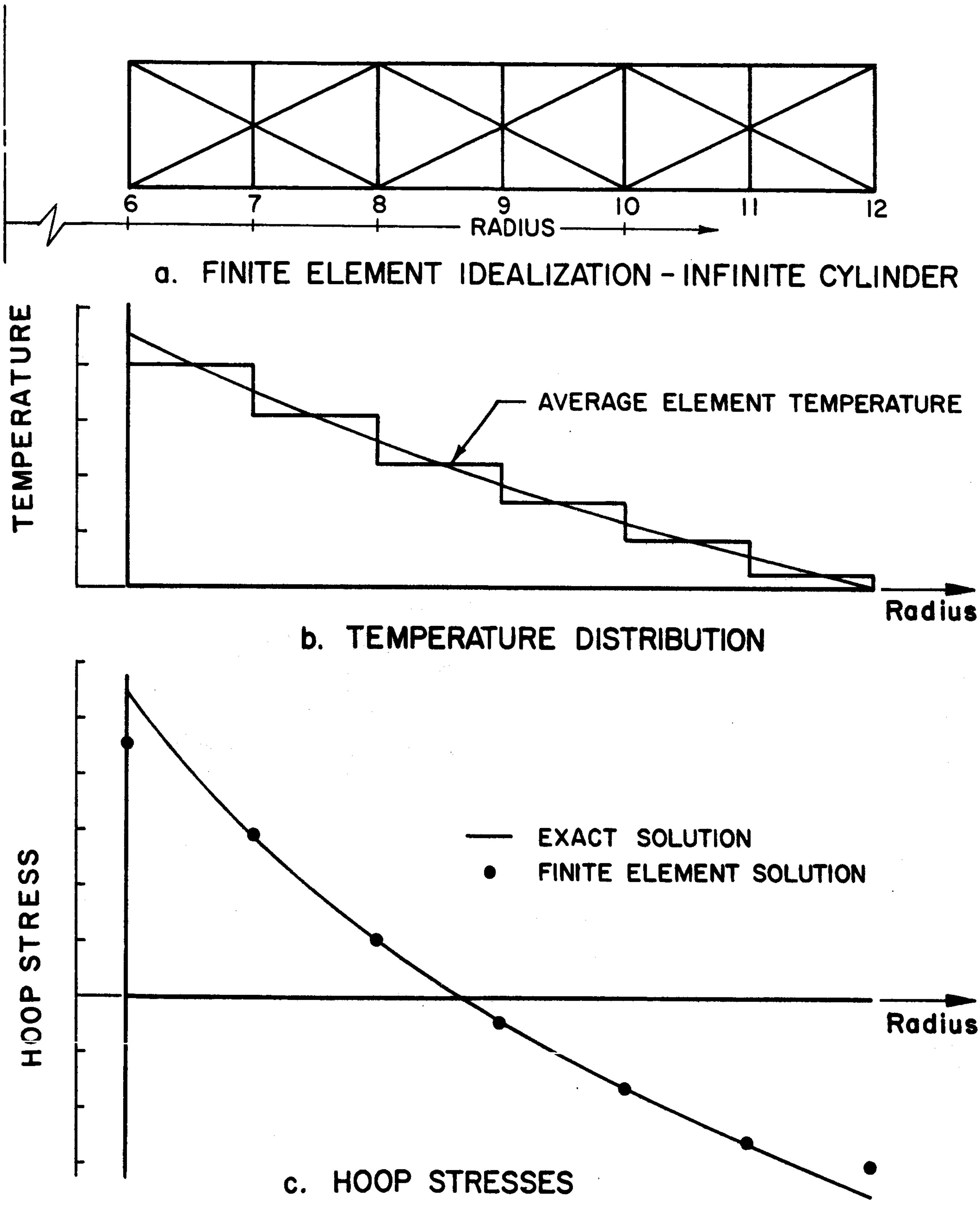


FIG. 2.5 ANALYSIS OF INFINITE CYLINDER

program, it is only necessary to select an element idealization of the structure and to supply the computer program with data that numerically define the system of elements. Therefore, the program may be used as a tool in design since changes in materials and geometry of the structures may involve only minor changes in the input data.

In addition, the program may be extended to include the effects of anisotropic materials. For this case, the input to the program must be expanded to include the general elastic constants defined by Eq. (2.5).

For the analysis of a specific type of structure, this program can be further automated by incorporating a mesh generator and the calculation of temperature-dependent material properties. In the next section of this report the program is specialized to the thermal stress analysis of axisymmetric heat shields for manned spacecraft.

PART III AUTOMATED PROGRAM FOR AXISYMMETRIC HEAT SHIELDS SUBJECTED TO AXISYMMETRIC LOADS

A. INTRODUCTION

The general computer program for the analysis of arbitrary axisymmetric structure, as given in the previous section, can be applied to the thermal stress analysis of heat shields. However, the use of this program for such a complex structure involves a large amount of detail work to select the finite element idealization and to prepare the computer input. In addition, the convergence of the Gauss-Seidel iteration procedure is slow for this type of structure and a solution may require an excessive amount of computer time.

By restricting the general computer program to the analysis of heat shields and by automating the input, a considerably more efficient program can be developed. The geometry of the heat shield is supplied to the computer program in the form of R and Z coordinates and ablator thickness at various points along the bond line. The required triangular mesh and the temperature at the grid points are generated within the program. Material properties at various temperatures are supplied in tabular form and the program automatically develops analytical expressions for the material properties by least square techniques. The flanges of the sandwich shell are idealized by special conical shell elements and the honeycomb core material is treated as a separate material. This approach eliminates the need for the establishing of a pseudo-thickness for the composite sandwich shell. The solution of the equilibrium equations, which was previously obtained by an iterative approach is accomplished by a direct solution procedure. Because of their significance, stresses within the sandwich plates and at the bond line are included in the computer output.

B. MESH GENERATION

A typical finite element idealization of the cross-section of a heat shield is shown in Figure 3.1. The basic element in this system is a quadrilateral ring, which in turn is composed of two triangular rings (Part II). In this particular case, the sandwich shell is represented by the first two rows of elements and the ablator is idealized by four rows of elements; there are 30 points in the meridional direction. The specific mesh configuration is a variable which is supplied to the computer program.

In general, the geometry of the shell is given by the R-Z coordinates of the points at the bond line between the sandwich shell and the ablator. The points on lines perpendicular to the bond line inside the variable thickness ablator and inside the constant thickness sandwich shell are generated automatically within the program. Thin shell cone elements are used to represent the face plates of the sandwich shell.

From a given temperature distribution at the bond line, the grid point temperatures are assumed to be constant within the shell and are assumed to vary parabolically within the ablator.

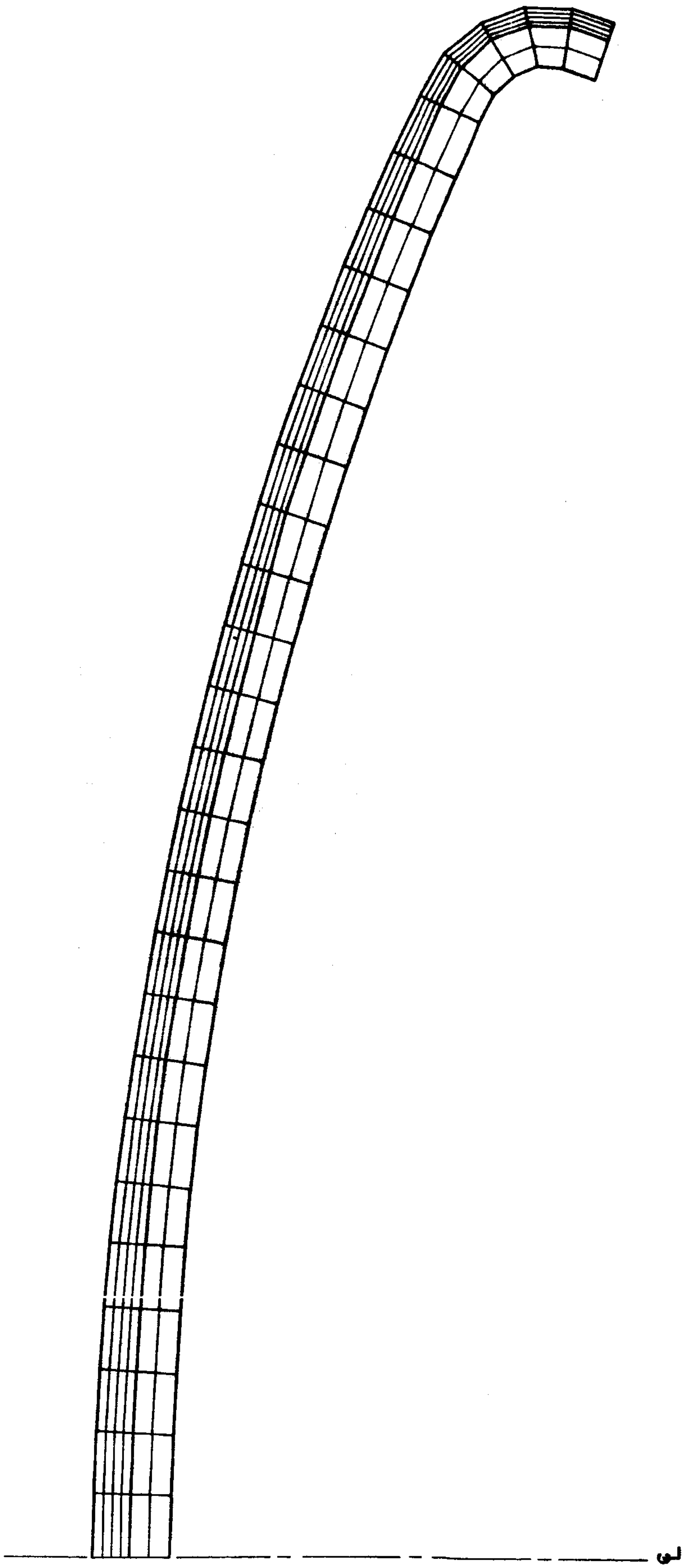


Fig. 3.1 Typical Finite Element Idealization of Heat Shield

C. DETERMINATION OF DISPLACEMENTS AND STRESSES

Based on temperature dependent material properties, the equilibrium relationship for each quadrilateral ring is developed and then combined to form the equilibrium equations of the complete system of rings. Similarly the stiffness properties of the face plates of the sandwich shell are incorporated into the equilibrium of the system. The axisymmetric behavior of a typical conical element is a special case of the non-axisymmetric behavior which is given in Appendix C. The unknowns in this set of equations are the vertical and radial displacements at each grid point in the system. The satisfying of possible displacement boundary conditions requires that these equations be modified as indicated by Equation (1.25). Because of the physical characteristics of the heat shield, the resulting set of equations is in band form. Appendix A indicates the necessary modification to restrict the standard Gaussian elimination procedure to the solution of symmetrical band systems. This approach results in a definite increase in capacity and speed over the iterative technique and it eliminates the problem of convergence.

After the equilibrium equations are solved for the unknown grid point displacements, the average stresses within each triangular ring are calculated as indicated in Part II of this report. Based on the stresses in the two triangular rings, average quadrilateral stresses are calculated for each quadrilateral ring in the system.

D. COMPUTER PROGRAM

The first step in the stress analysis of an axisymmetric heat shield is to select points at the bond line at regular intervals along the meridian of the shield. A quadrilateral mesh is automatically developed by the program from the R and Z coordinates. The material properties vs temperature for the ablator and bond are supplied to the computer in tabular form and the computer program automatically determines analytical expressions for the properties by a least square procedure (see Appendix B for details of method). Finally, the grid points which are to be restrained and the external loads which act at grid points are specified.

1. Input Information

The following sequence of punched cards numerically defines the heat shield to be analyzed.

a. FIRST CARD - (72H)

Columns 1 to 72 of this card contains information to be printed with results

b. SECOND CARD - (6I5, 2F10.2)

Columns 1 - 5 Number of points along the shield - NMAX
6 - 10 Number of points thru the thickness-MMAX
11 - 15 Location of bond line - MB
16 - 20 Number of material property cards - NP

21 - 25 Number of points with radial and axial loads - NL

26 - 30 Number of additional boundary conditions - NB

31 - 40 Surface temperature of ablator

41 - 50 Zero stress temperature

c. THIRD CARD - Properties of Sandwich Core (4F10.2)

Columns 1 - 10 Modulus of elasticity

11 - 20 Poisson's ratio

21 - 30 Coefficient of thermal expansion

31 - 40 Thickness of core

d. FOURTH CARD - Properties of Sandwich Face Plates (4F12.2)

Columns 1 - 10 Modulus of elasticity

11 - 20 Poisson's ratio

21 - 30 Coefficient of thermal expansion

31 - 40 Thickness of single face plate

e. GEOMETRY CARDS - (4F10.2)

One card per point along shield, in order from axis of symmetry to edge (NMAX cards).

Columns 1 - 10 R-ordinate at bond line

11 - 20 Z-ordinate at bond line

21 - 30 Temperature at bond line

31 - 40 Normal thickness of ablator

f. MATERIAL PROPERTY CARDS - (4F10.2)

One card for each temperature (NP cards)

Columns 1 - 10 Temperature

11 - 20 Modulus of elasticity of ablative material

21 - 30 Modulus of elasticity of bond material

31 - 40 Coefficient of thermal expansion for ablator and bond materials

g. LOAD CARDS - (2I5, 2F10.2)

One card for each point which is loaded externally (NL cards). N and M specify the grid location of the point.

Columns 1 - 5 N(Meridional direction)

6 - 10 M (Thickness direction)

11 - 20 Radial Load
21 - 30 Axial Load
} Total load acting on one radian segment

h. BOUNDARY CONDITION CARDS - (3I5)

One card for each point which is restrained (NB cards). N and M specify the grid location of the point.

Columns 1 - 5 N (Meridional direction)

6 - 10 M (Thickness direction)

11 - 15 Boundary Code

Code = 1 point fixed in R-direction

Code = 2 point fixed in Z-direction

Code = 3 point fixed in both the R and Z-directions

2. Output Information

The following information is generated and printed by the computer program:

a. Input data

b. Least squares evaluation of the temperature dependent material property data

c. Coordinates and temperatures of all grid points

d. R and Z displacement at all grid points

e. Average stresses in quadrilateral rings

f. Stresses in sandwich face plates

g. Stresses in bond layer

3. Timing

The computer time required by this program for an axisymmetric analysis of a heat shield is approximately

$$\text{time} = A + B \cdot (\text{NMAX}) \cdot (\text{MMAX})^2 \text{ (seconds)}$$

The constants A and B depend on the specific computer system which is employed. For the IBM 7094 A=20 and B=0.02, and the time required for a 30 x 7 mesh is 50 seconds.

4. Program Listing

A card listing of the FORTRAN II source deck for the automated computer program for the axisymmetric stress analysis of heat shields is given in Appendix E. The program is compiled for a maximum grid size of 40 points in the meridional direction and 10 points through the thickness. Material properties can be specified by a maximum of 50 cards.

E. EXAMPLES

Several axisymmetric analyses of a heat shield were conducted to evaluate the significance of the various structural parameters. A typical finite element idealization of the heat shield is shown in Figure 3.1.

1. Effect of Mesh Size

The first example was selected to illustrate the effect of mesh size on the accuracy of the displacements and stresses developed within the heat shield. For a structure fixed at the edge, typical results of two analyses with different meshes are shown in Figure 3.2. This example illustrates that two layers of elements in the sandwich shell are adequate for the purposes of predicting stresses. It is of interest to note that the stress distribution varies linearly within the sandwich shell, thereby confirming the assumption made in thin shell theory. The displacements for these two analyses differed by less than one percent.

2. Effect of Ablator Thickness on Stress Distribution

Figure 3.3 shows typical results of three analyses of heat shields with different ablator thicknesses. In general, the magnitude of

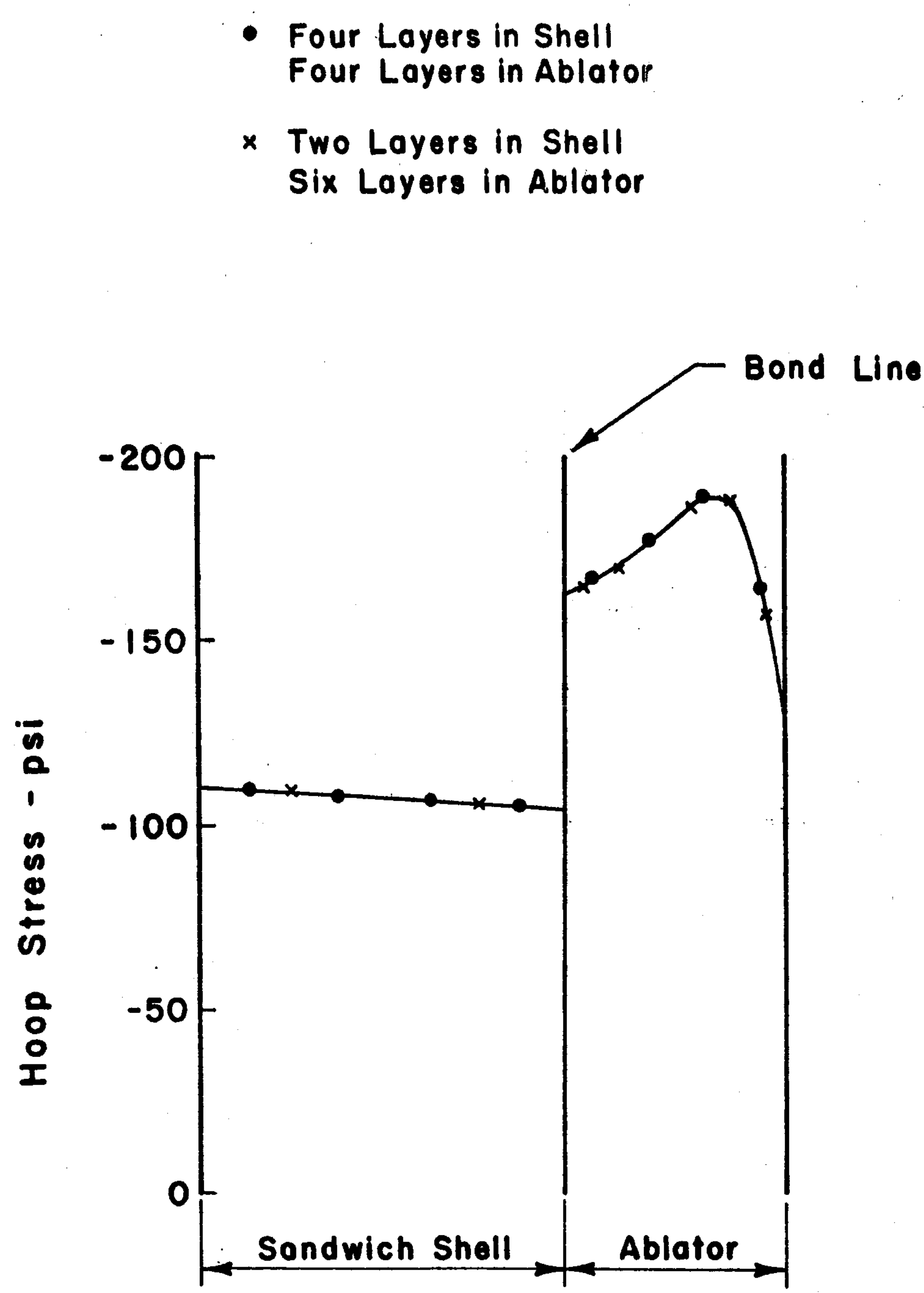


Fig.3.2 Effect of Mesh Size on Stress Distribution

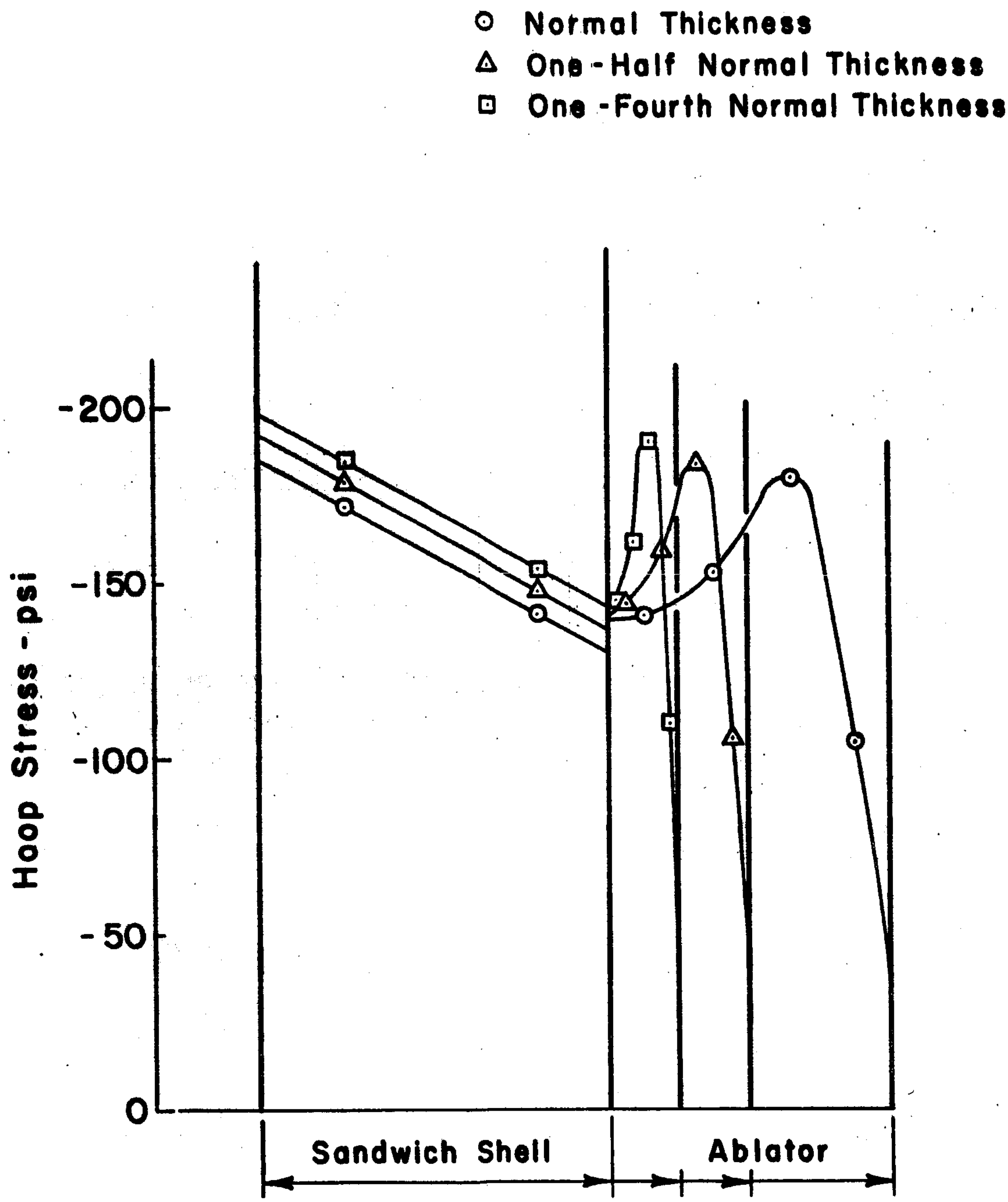


Fig. 3.3 Effect of Thickness of Ablator on Stress Distribution

the maximum stresses in the ablator were in good agreement. This example illustrates that the thickness of the ablator is not an important structural parameter at the temperature of re-entry.

3. Effect of Boundary Conditions on the Behavior of the Heat Shield

The support condition which is imposed on the heat shield is an extremely important parameter. Figure 3.4 illustrates the deflected position of the bond line for two different support conditions. The resulting stresses differ significantly. Therefore, it is important that the boundary condition which is imposed on the finite element system be a realistic approximation of the physical support condition which exists in the actual heat shield.

F. DISCUSSION

The automated computer program presented in this section reduces the analysis of an arbitrary heat shield subjected to axisymmetric thermal or mechanical loads to a simple procedure. The program automatically generates the finite element grid, evaluates temperature-dependent material properties, solves the equilibrium equations for the grid point displacements and calculates stresses within elements, sandwich shell face plates and at the bond layer. Arbitrary boundary conditions can be imposed since any of the grid points may be restrained in either the R or Z directions.

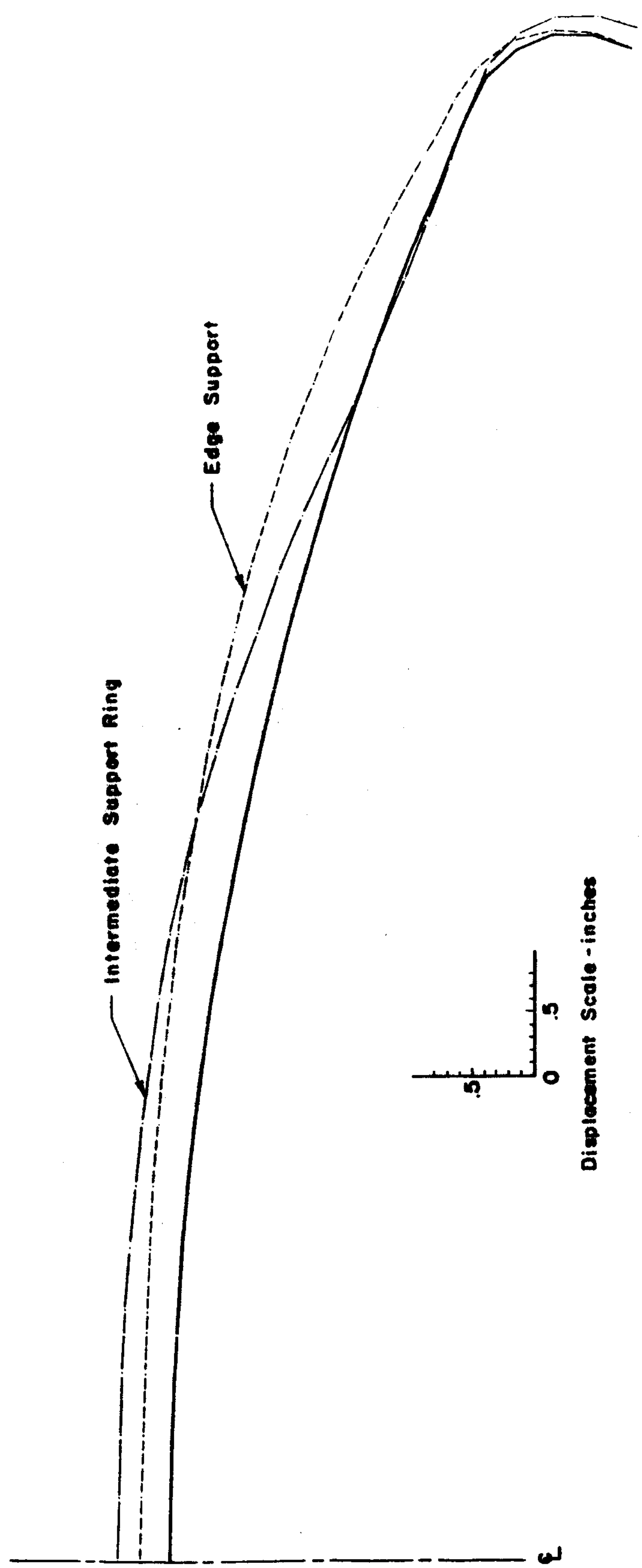


Fig. 3.4 Effect of Boundary Conditions on Deflected Shape (Bond Line)

PART IV: AUTOMATED PROGRAM FOR AXISYMMETRIC HEAT SHIELDS SUBJECTED TO NON-AXISYMMETRIC LOADS

A. INTRODUCTION

In general, the heat shield of a manned spacecraft is composed of a constant thickness sandwich shell and an ablator which varies in thickness in both the meridional and circumferential directions. The temperature distribution experienced by the heat shield is also non-axisymmetric. Because the ablator, at high temperatures, is not a major structural element, contributing to the overall behavior of the heat shield, an approximation of its properties in the circumferential direction is justified. The approximation, that it is axisymmetric, reduces the stress analysis of a non-axisymmetric heat shield to the stress analysis of an axisymmetric structure subjected to non-axisymmetric thermal loads. This involves the expansion of the temperature distribution and the final displacements of the system in Fourier series. By making use of the orthogonality properties of the harmonic functions the three-dimensional analysis is divided into a series of uncoupled two-dimensional analyses.

B. THEORY FOR THE ANALYSIS OF AN AXISYMMETRIC BODY SUBJECTED TO NON-AXISYMMETRIC LOADS

A theory is presented for the analysis of solids of revolution subjected to non-axisymmetric loads which are symmetric about a plane containing the axis of revolution. Figure 4.1, a view of a plane perpendicular to the axis of revolution, shows the trace of the plane of symmetry. Anisotropic material properties, which are constant along any circumferential line, are included in this formulation.

The structure is idealized as a series of rings with triangular cross-sections; the rings are interconnected at their nodal circles, i.e., at the circles containing the vertices of the triangles, (Figure 4.2). Loads acting on the structure are replaced by statically equivalent concentrated forces acting along the nodal circles.

1. Strain-Displacement Relationship

By noting the axisymmetry of the geometry and the material properties of the body and the plane of symmetry for deformations, the displacements in r, θ, z coordinates may be written in the following form:

$$u_r = \sum u_{rn}(r,z) \cos n\theta \tag{4.1a}$$

$$u_\theta = \sum u_{zn}(r,z) \sin n\theta \tag{4.1b}$$

$$u_z = \sum u_{\theta n}(r,z) \cos n\theta \tag{4.1c}$$

Within each ring element the r and z variation of the Fourier coefficients of the displacements are assumed to be linear, i.e.,

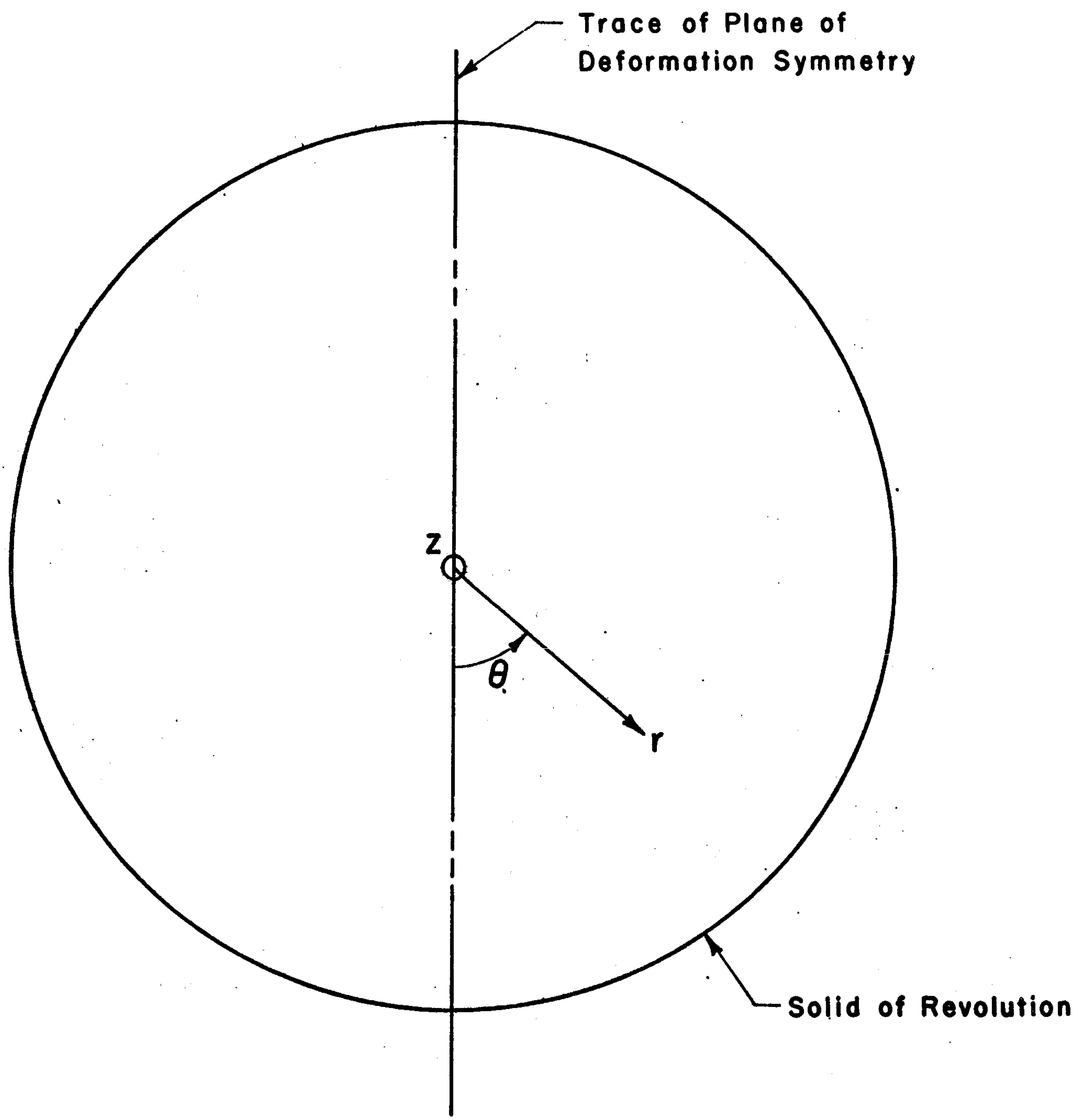


Fig. 4.1 Cylindrical Coordinate System
Embedded in a Solid of Revolution

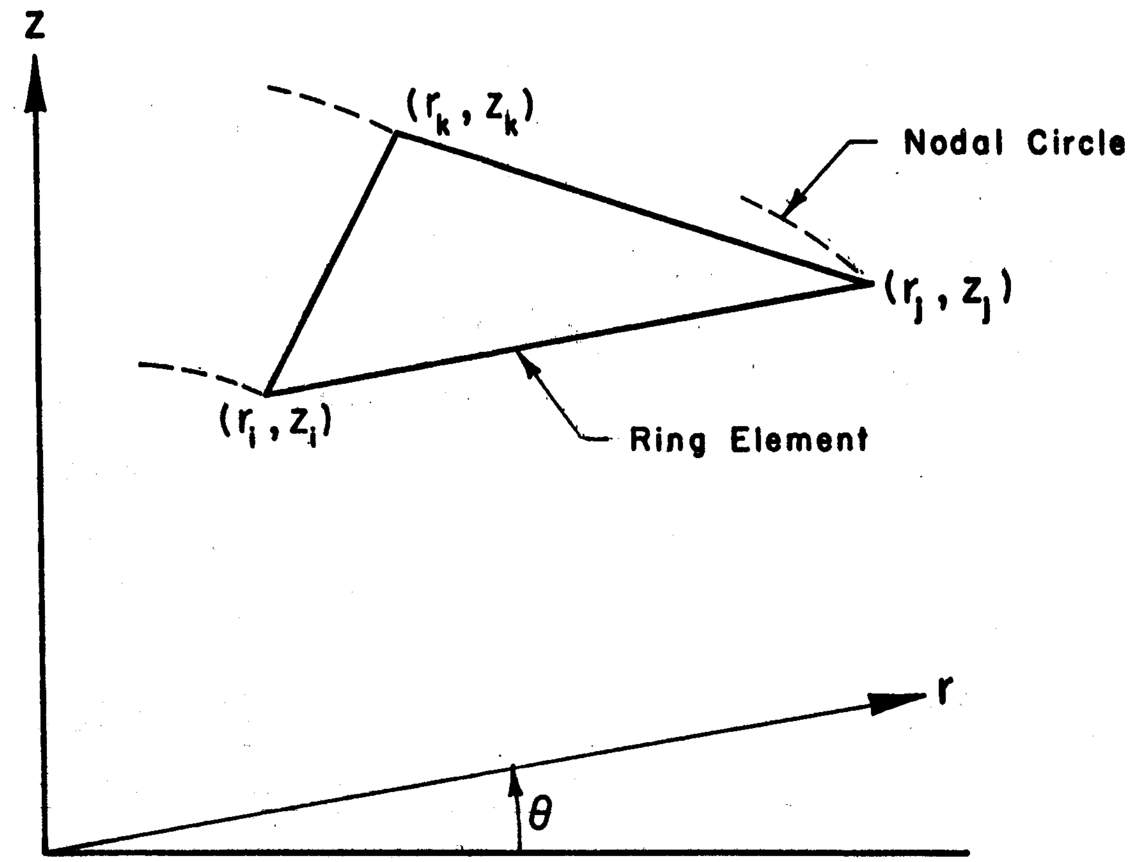


Fig. 4.2 Cross Section of a Ring Element

$$u_{rn} \approx k_{1n} + k_{2n}\, r + k_{3n} z \qquad (4.2a)$$

$$u_{\theta n} \approx k_{4n} + k_{5n}\, r + k_{6n} z \qquad (4.2b)$$

$$u_{zn} \approx k_{7n} + k_{8n}\, r + k_{9n} z \qquad (4.2c)$$

Now expressing the constants k_{mn} in terms of the corner values of the Fourier coefficients of the displacements, i.e., in terms of

u^i_{rn}, $u^i_{\theta n}$, u^i_{zn}, u^j_{rn}, $u^j_{\theta n}$, u^j_{zn}, u^k_{rn}, $u^k_{\theta n}$ and u^k_{zn}

$$\begin{bmatrix} k_{1n} & k_{4n} & k_{7n} \\ k_{2n} & k_{5n} & k_{8n} \\ k_{3n} & k_{6n} & k_{9n} \end{bmatrix} = \begin{bmatrix} T \end{bmatrix} \begin{bmatrix} u^i_{rn} & u^i_{\theta n} & u^i_{zn} \\ u^j_{rn} & u^j_{\theta n} & u^j_{zn} \\ u^k_{rn} & u^k_{\theta n} & u^k_{zn} \end{bmatrix} \qquad (4.3a)$$

with

$$\begin{bmatrix} T \end{bmatrix} = \frac{1}{D} \begin{bmatrix} r_j z_k - z_j r_k & r_k z_i - r_i z_k & r_i z_j - r_j z_i \\ z_j - z_k & z_k - z_i & z_i - z_j \\ r_k - r_j & r_i - r_k & r_j - r_i \end{bmatrix} \qquad (4.3b)$$

and $$D = r_j(z_k - z_i) + r_i(z_j - z_k) + r_k(z_i - z_j) \qquad (4.3c)$$

Combining Equation (4.1) with the strain-displacement relationships, the following expressions for the strains are found:

$$\epsilon_r = \frac{\partial u_r}{\partial r} = \sum \epsilon_{rn} \cos n\theta \qquad (4.4a)$$

$$\epsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \sum \epsilon_{\theta n} \cos n\theta \tag{4.4b}$$

$$\epsilon_z = \frac{\partial u_z}{\partial z} = \sum \epsilon_{zn} \cos n\theta \tag{4.4c}$$

$$\gamma_{r\theta} = \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = \sum \gamma_{r\theta n} \sin n\theta \tag{4.4d}$$

$$\gamma_{rz} = \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) = \sum \gamma_{rzn} \cos n\theta \tag{4.4e}$$

$$\gamma_{\theta z} = \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) = \sum \gamma_{\theta zn} \sin n\theta \tag{4.4f}$$

where

$$\epsilon_{rn} = \frac{\partial u_{rn}}{\partial r} \tag{4.5a}$$

$$\epsilon_{\theta n} = \frac{1}{r}(nu_{\theta n} + u_r) \tag{4.5b}$$

$$\epsilon_{zn} = \frac{\partial u_{zn}}{\partial z} \tag{4.5c}$$

$$\gamma_{r\theta n} = \frac{\partial u_{\theta n}}{\partial r} - \frac{u_{\theta n}}{r} - \frac{nu_{rn}}{r} \tag{4.5d}$$

$$\gamma_{rzn} = \frac{\partial u_{zn}}{\partial r} + \frac{\partial u_{rn}}{\partial z} \tag{4.5e}$$

$$\gamma_{\theta zn} = \frac{\partial u_{\theta n}}{\partial z} - n\frac{u_{zn}}{r} \tag{4.5f}$$

Within a given ring the approximate values of strain for the harmonic n are calculated by combining Equations (4.2), (4.3) and (4.5). The hoop strain is assumed to be constant within the ring and $\frac{u_{rn}}{r}$ is approximated by

$$\frac{u^i_{rn}}{3\bar{r}} + \frac{u^j_{rn}}{3\bar{r}} + \frac{u^k_{rn}}{3\bar{r}}, \quad \text{with} \quad \bar{r} = \frac{1}{3}(r_i + r_j + r_k)$$

Thus, for the harmonic n the six components of strain within the element are given in terms of the nine corner displacements by the following matrix equation:

$$\left[\epsilon_n\right] = \left[G_n\right]\left[u_n\right] \tag{4.6a}$$

The strain-displacement transformation matrix (for convenience it is written in its transposed form) is defined on the next page, Eq. (4.6b).

2. <u>Stress-Strain Relationship</u>

The stress-strain relationship for the harmonic n is written in the following symbollic form:

$$\left[\sigma_n\right] = \left[C\right]\left[\epsilon_n\right] + \left[\tau_n\right] \tag{4.7a}$$

For an isotropic material this becomes

$$\begin{bmatrix} \sigma_{rn} \\ \sigma_{\theta n} \\ \sigma_{zn} \\ \sigma_{r\theta n} \\ \sigma_{rzn} \\ \sigma_{\theta zn} \end{bmatrix} = \begin{bmatrix} \bar{\alpha} & \bar{\beta} & \bar{\beta} & 0 & 0 & 0 \\ \bar{\beta} & \bar{\alpha} & \bar{\beta} & 0 & 0 & 0 \\ \bar{\beta} & \bar{\beta} & \bar{\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \epsilon_{rn} \\ \epsilon_{\theta n} \\ \epsilon_{zn} \\ \epsilon_{r\theta n} \\ \epsilon_{rzn} \\ \epsilon_{\theta zn} \end{bmatrix} + \begin{bmatrix} \tau_n \\ \tau_n \\ \tau_n \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.7b}$$

$$
[G_n]^T = \begin{bmatrix}
\frac{z_j-z_k}{D} & \frac{1}{3\bar{r}} & 0 & -\frac{n}{3\bar{r}} & \frac{r_k-r_j}{D} & 0 \\
\frac{z_k-z_i}{D} & \frac{1}{3\bar{r}} & 0 & -\frac{n}{3\bar{r}} & \frac{r_i-r_k}{D} & 0 \\
\frac{z_i-z_j}{D} & \frac{1}{3\bar{r}} & 0 & -\frac{n}{3\bar{r}} & \frac{r_j-r_i}{D} & 0 \\
0 & \frac{1}{3\bar{r}} & 0 & (\frac{z_j-z_k}{D} - \frac{1}{3\bar{r}}) & 0 & \frac{r_k-r_j}{D} \\
0 & \frac{n}{3\bar{r}} & 0 & (\frac{z_j-z_k}{D} - \frac{1}{3\bar{r}}) & 0 & \frac{r_k-r_j}{D} \\
0 & \frac{n}{3\bar{r}} & 0 & (\frac{z_k-z_i}{D} - \frac{1}{3\bar{r}}) & 0 & \frac{r_i-r_k}{D} \\
0 & 0 & \frac{r_k-r_j}{D} & 0 & \frac{z_j-z_k}{D} & -\frac{n}{3\bar{r}} \\
0 & 0 & \frac{r_i-r_k}{D} & 0 & \frac{z_k-z_i}{D} & -\frac{n}{3\bar{r}} \\
0 & 0 & \frac{r_j-r_i}{D} & 0 & \frac{z_i-z_j}{D} & -\frac{n}{3\bar{r}}
\end{bmatrix} \quad (4.6b)
$$

where
$$\bar{\alpha} = \frac{(1-\nu)\ E}{(1+\nu)(1-2\nu)} \qquad (4.8a)$$

$$\bar{\beta} = \frac{\nu \quad E}{(1+\nu)(1-2\nu)} \qquad (4.8b)$$

$$\mu = \frac{E}{2(1+\nu)} \qquad (4.8c)$$

$$\tau_n = - \frac{E\alpha}{(1-2\nu)}\ T_n \qquad (4.8d)$$

T_n is the Fourier coefficient for the expansion of the average temperature change within the ring

$$T = \sum T_n \cos n\theta \qquad (4.9)$$

3. <u>Equilibrium Equation for Harmonic n</u>

By recognizing the othogonality of the harmonic functions the same procedure which was used in Part I may be used to develop the equilibrium equations for an element subjected to harmonic loading. Therefore, Equation (1.15) is rewritten as

$$\left[S_n\right] = \left[k_n\right]\left[u_n\right] + \left[L_n\right] \qquad (4.10)$$

where
$$\left[k_n\right] = \int \left[G_n\right]^T\left[C\right]\left[G_n\right] dV \qquad (4.11)$$

$$\left[L_n\right] = \int \left[G_n\right]^T\left[\tau_n\right] dV \qquad (4.12)$$

Within a ring the matrices $\left[G_n\right]$ and $\left[C\right]$ are not a function of space; therefore, Equations (4.11) and (4.12) reduce to

$$\left[k_n\right] = V. \left[G_n\right]^T \left[C\right] \left[G_n\right] \tag{4.13}$$

$$\left[L_n\right] = V. \left[G_n\right]^T \left[\tau_n\right] \tag{4.14}$$

where the volume V is given by Equation (2.10). Equilibrium of the overall structure requires that the sum of the nodal circle forces for all rings with a common nodal circle must equal the applied nodal force. This results in an equation of the following form for each harmonic:

$$\left[K_n\right] \left[r_n\right] = \left[R_n\right] \tag{4.15}$$

where the displacement vector $\left[r_n\right]$ contains all the displacement amplitudes u_{rn}, $u_{\theta n}$ and u_{zn} for all nodal circles in the system.

The equilibrium of the face plates is incorporated by a similar procedure. Appendix C gives the details of this development.

4. Determination of Displacements and Stresses

The number of harmonics required to represent the three-dimensional temperature distribution indicates the number of two-dimensional problems which must be solved. For each harmonic Equation (4.15) must be solved for the unknown displacement amplitudes. The corresponding strain amplitudes are calculated for each finite element by Equation (4.6) and then stress amplitudes are found by the application of Equation (4.7). The final displacements of the system for any angle are calculated from Equation (4.1). The final stresses are determined from the stress amplitudes by the following equations:

$$\sigma_r(r,z,\theta) = \sum \sigma_{rn} \cos n\theta \tag{4.16a}$$

$$\sigma_z(r,z,\theta) = \sum \sigma_{zn} \cos n\theta \tag{4.16b}$$

$$\sigma_\theta(r,z,\theta) = \sum \sigma_{\theta n} \sin n\theta \tag{4.16c}$$

$$\sigma_{rz}(r,z,\theta) = \sum \sigma_{rzn} \cos n\theta \tag{4.16d}$$

$$\sigma_{r\theta}(r,z,\theta) = \sum \sigma_{r\theta n} \sin n\theta \tag{4.16e}$$

$$\sigma_{\theta z}(r,z,\theta) = \sum \sigma_{\theta zn} \sin n\theta \tag{4.16f}$$

C. COMPUTER PROGRAM

The use of the non-axisymmetric heat shield program is similar to the axisymmetric program (Part III). The only additional input required is the three-dimensional temperature distribution. The computer program automatically develops the necessary Fourier coefficients for the temperature distribution and sums the series of two-dimensional analyses to produce the final displacements and stresses in the system.

1. Input Information

The following sequence of punched cards numerically defines the heat shield to be analyzed:

a. FIRST CARD - (72H)

Columns 1 to 72 of this card contains information to be printed with results

b. SECOND CARD - (6I5, 2F10.2)

Columns	1 - 5	Number of points along meridian of shield - NMAX
	6 - 10	Number of points through thickness - MMAX
	11 - 15	Location of bond line - MB
	16 - 20	Number of material property cards - NP
	21 - 25	Number of harmonic to be used in analysis - NL
	26 - 30	Number of boundary condition cards - NB
	31 - 40	Surface temperature of ablator
	41 - 50	Temperature of zero stress

c. THIRD CARD - Properties of Sandwich Core (4F10.2)

Columns	1 - 10	Modulus of elasticity
	11 - 20	Poisson's ratio
	21 - 30	Coefficient of thermal expansion
	31 - 40	Thickness of core

d. FOURTH CARD - Properties of Sandwich Fact Plates (4F10.2)

Columns	1 - 10	Modulus of elasticity
	11 - 20	Poisson's ratio
	21 - 30	Coefficient of thermal expansion
	31 - 40	Thickness of single face plate

e. GEOMETRY CARDS - (4F10.2)

One card per point along shield in order from axis of symmetry to edge (NMAX cards).

Columns 1 - 10 R-ordinate at bond line

11 - 20 Z-ordinate at bond line

21 - 30 Temperature at bond line

31 - 40 Normal thickness of ablator

The temperature information is used by the program to determine the axisymmetric temperature-dependent material properties.

f. MATERIAL PROPERTY CARDS - (4F10.2)

One card for each temperature (NP cards)

Columns 1 - 10 Temperature

11 - 20 Modulus of elasticity of ablative material

21 - 30 Modulus of elasticity of bond material

31 - 41 Coefficient of thermal expansion for ablative and bond materials

g. THREE-DIMENSIONAL TEMPERATURE DISTRIBUTION CARDS

A table of bond line temperature values at 10 degree increments along 9 circumferential lines is punched in the following form:

1st. card - (9F8.0)

R-ordinates of 9 circumferential points on bond line

2nd card - (9F8.0)

Z-ordinates of 9 circumferential points on bond line

3rd to 21st card - (9F8.0)

One card for each 10 degree increment (0 to 180^{o}). Each card contains the 9 temperatures which correspond to the above R and Z-ordinates.

h. BOUNDARY CONDITION CARDS - (3I5)

One card per restrained nodal circle (NB cards)

Column 1 - 5 N } mesh point N, M
6 - 10 M

11 - 15 = 1 restrained in R-direction
2 restrained in θ-direction
3 restrained in Z-direction
4 restrained in R and θ-directions
5 restrained in R and Z-directions
6 restrained in θ and Z-directions
7 restrained in R, θ and Z directions

i. PRINT ANGLE CARDS - (1F5.0)

Column 1 - 5 angle θ

For each "print angle card" a complete set of displacement and stresses are printed for angle θ.

2. <u>Output Information</u>

The following information is generated and printed by the computer program:

a. Input data

b. Least squares evaluation of the temperature-dependent material property data

c. Coordinates and temperature of all grid points

d. Two-dimensional Fourier temperature coefficients

e. For each print angle

(1) R, Z, and θ displacement at all grid points
(2) Average stresses in quadrilatoral rings
(3) Stresses in sandwich face plates

3. Timing

The computer time required by this program for the non-axisymmetric analysis of a heat shield is approximately

$$\text{time} = A + B \cdot (\text{NMAX}) \cdot (\text{MMAX})^2 \cdot \text{NL} \text{ (seconds)}$$

For the IBM 7094 computer A=20 and B=0.05, and the time required for a 30 x 7 mesh with 4 harmonics is approximately 5 minutes.

4. Program Listing

A card listing of the FORTRAN II source deck for the computer program for the non-axisymmetric analysis of heat shields is given in Appendix F. The program is compiled for a maximum grid size of 30 points in the meridional direction and 8 points through the thickness. A maximum of 10 harmonics may be considered. Material properties can be specified by a maximum of 50 cards.

Standard input tape 5 and output tape 6 are used by the program. Tape 20 is used for temporary storage within the program; it may be necessary to change this tape unit to conform with local computer center policy.

D. EXAMPLES

Two analyses of axisymmetric heat shields subjected to non-axisymmetric temperature distribution were conducted to illustrate the application of the program. In both cases, the axisymmetric finite element mesh was similar to the mesh given by Figure 3.1. For the purpose of reference the station layout along the bond line is shown in Figure 4.3. The three-dimensional bond layer temperature and ablator thickness distribution for angles $\theta = 0, 90^{\circ}, 180^{\circ}$ are plotted in Figure 4.4.

In Analysis A the axisymmetric properties of the ablator are assumed to be equal to the properties of the actual ablator at $\theta = 0$. In Analysis B the ablator properties at $\theta = 180^{\circ}$ are used. For both analyses the surface temperature of the ablator is 1000 F and the temperature at the bond surface is given by Figure 4.5. Station 20 (Figure 4.3) is restrained at the inside surface of the shield to simulate the effect of an intermediate support ring.

The computer output for a non-axisymmetric analysis contains displacements and stresses at many points in the heat shield; however, only the typical results are presented. For Analysis A the deflected shape of the bond line at three sections is plotted in Figure 4.6a. The non-axisymmetric behavior is significant. The displacements from Analysis B are shown in Figure 4.6b; they are essentially the same as those found by Analysis A. This again indicates that the ablator's thickness and property variations in the circumferential direction are not of major importance at these temperatures. Hoop stresses at station 15 are

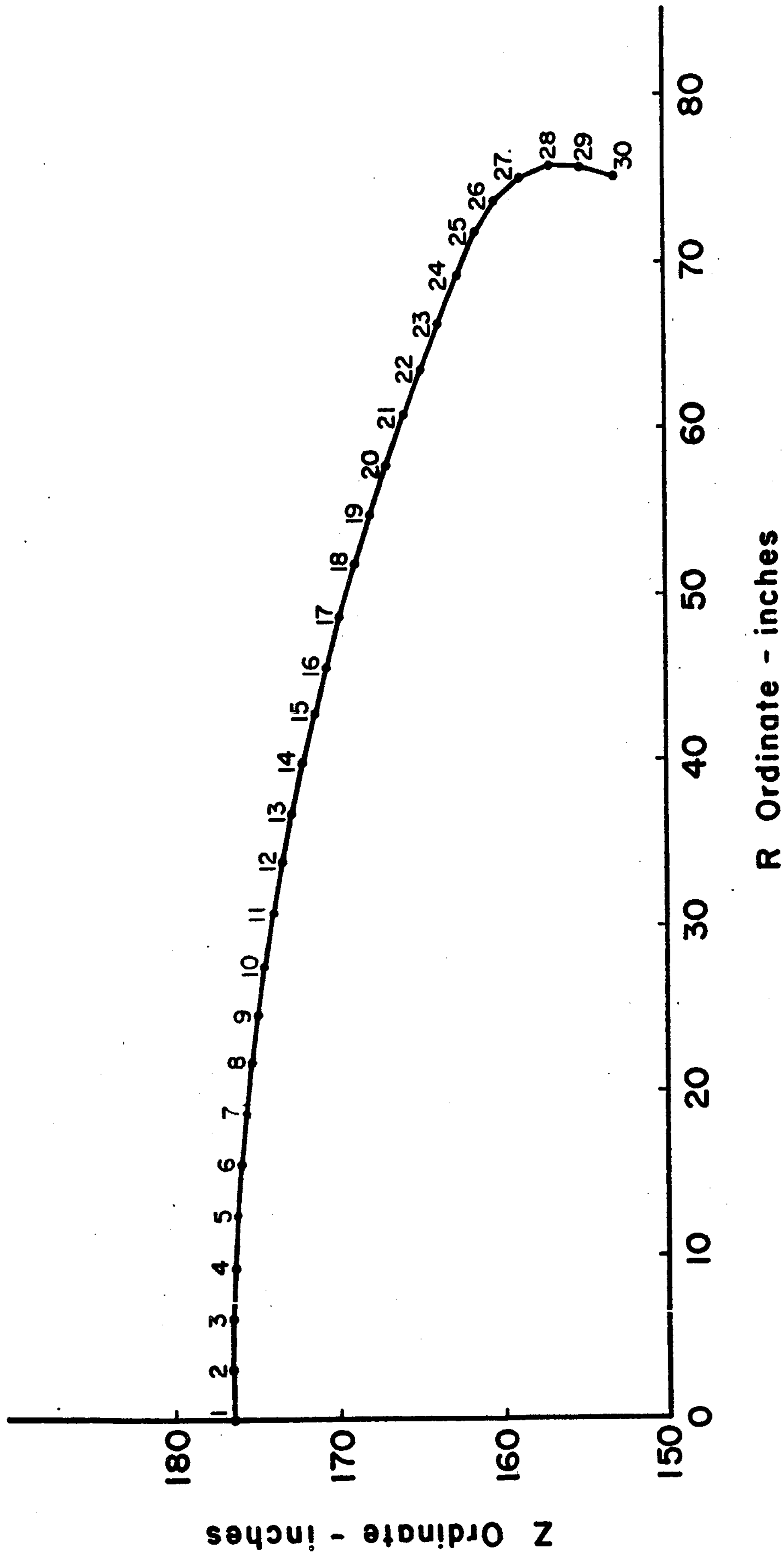


Fig. 4.3 Station along Bond Line

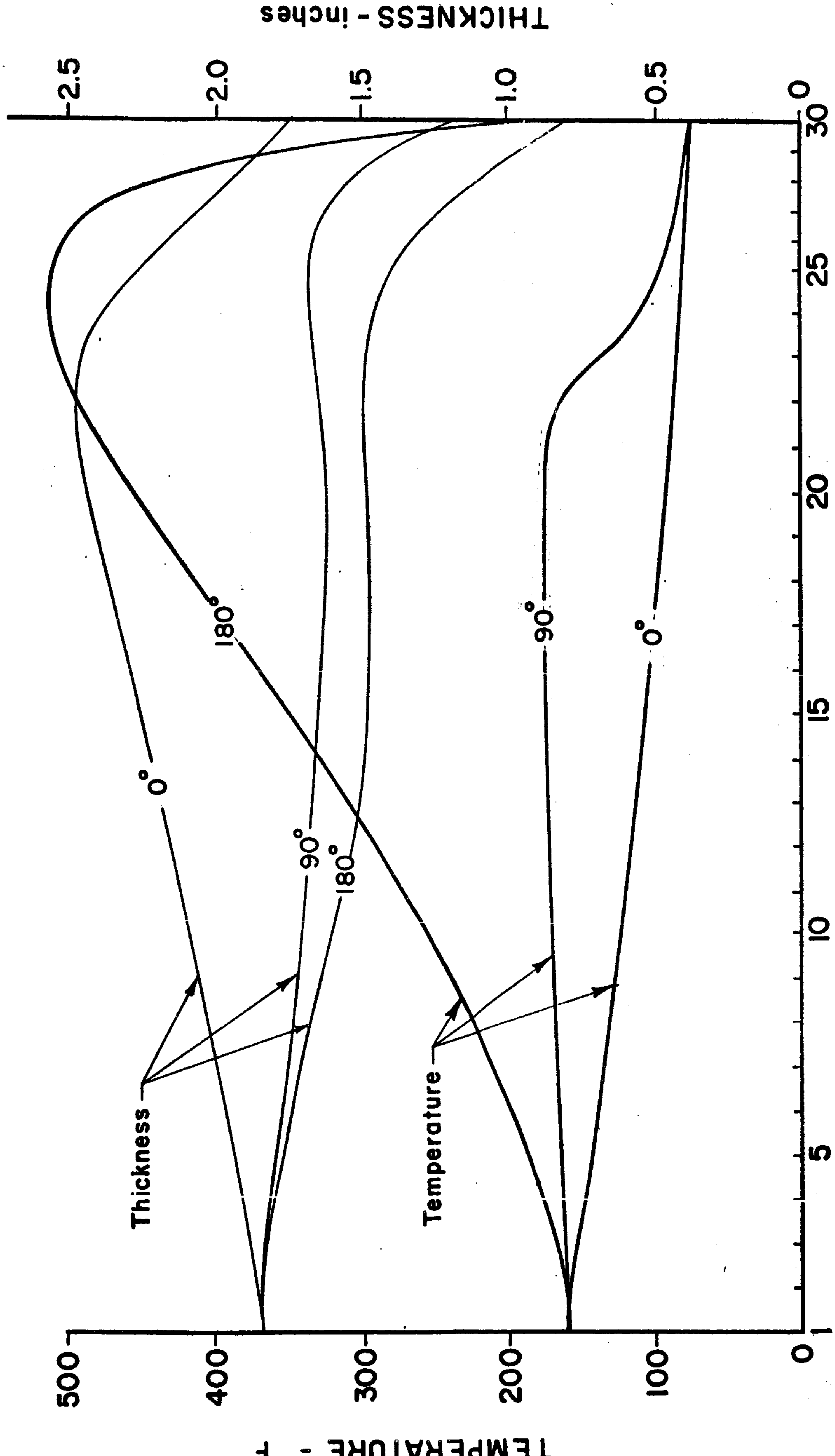


Fig. 4.4 Bond Layer Temperature and Ablator Thickness Distribution

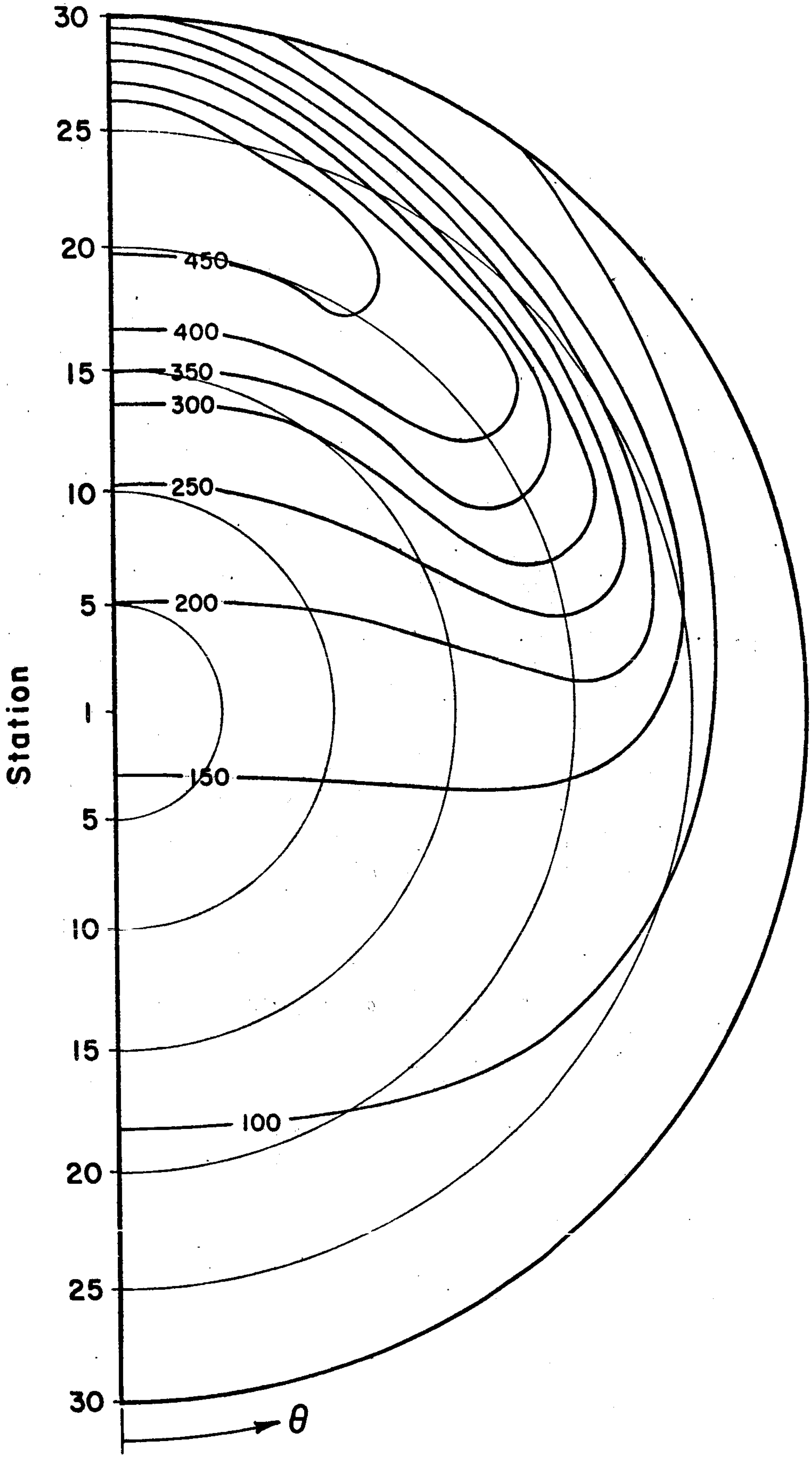


Fig. 4.5 Temperature at Bond Surface

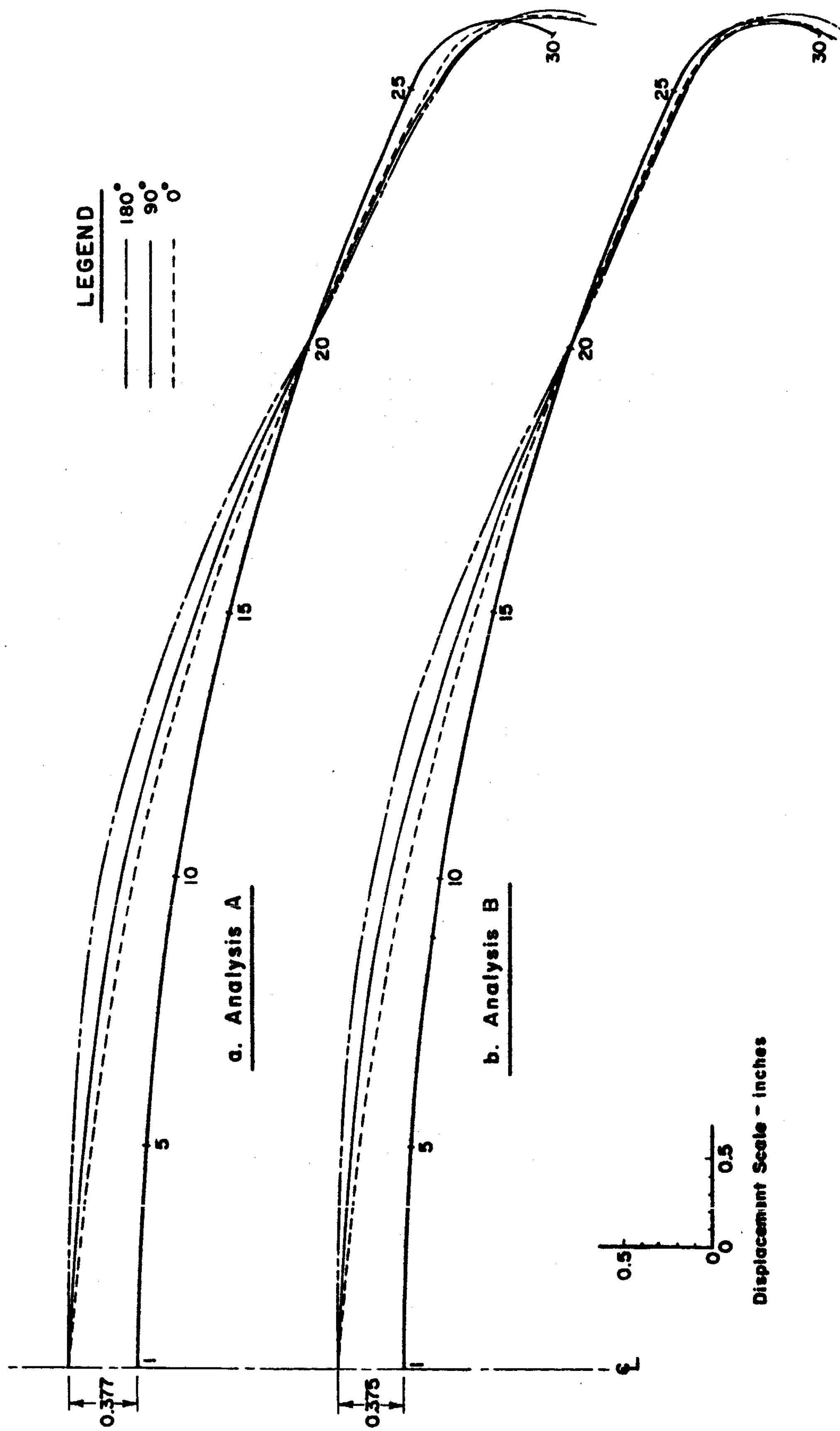


Fig. 4.6 Deflected Shape at Bond Surface

plotted in Figure 4.7 for two values of θ. In both analyses the stresses within the sandwich shell are in fair agreement since the material properties do not change at these temperatures. However, within the ablator, where the material properties are strongly temperature-dependent, the stresses differ significantly. Of course, the particular solution is most reasonable if the assumed material properties correspond with those at the position of the desired stress. Hence, for $\theta = 0$, Analysis A is considered the best approximation and similarly for $\theta = 180^\circ$, Analysis B is the most reasonable.

E. DISCUSSION

In this section a method and the resulting computer program are presented for the analysis of axisymmetric heat shields subjected to non-axisymmetric thermal loads. The program may be used to analyze an approximate solution to a non-axisymmetric heat shield if a number of solutions are obtained and then are judicially evaluated. At the section where the assumed axisymmetric ablator properties (temperature and thickness) correspond to the local properties, the stresses will be a good approximation of the actual state of stress in the non-axisymmetric heat shield. Therefore, for each angle θ for which stresses are desired a separate structure must be evaluated.

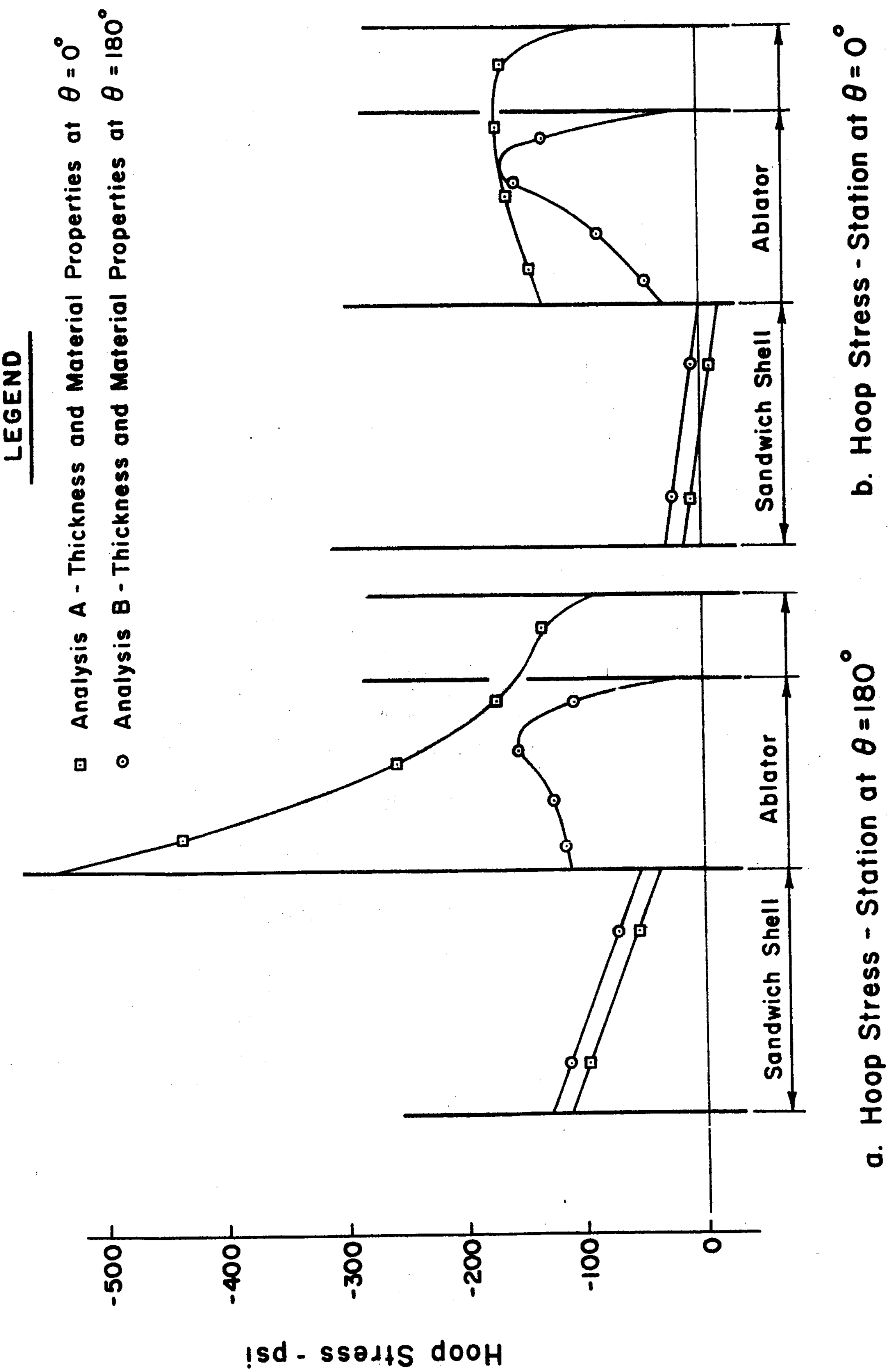


Fig. 4.7 Typical Results of Non-Axisymmetric Heat Shield Analysis

It should be pointed out that the computer program can be extended to include non-axisymmetric pressure loading, displacement boundary condiditons and the effects of elastic supports. However, this additional investigation was beyond the scope of the present effort.

F. COLD SOAK CONDITION

A necessary approximation of the method of analysis which is presented in this report is that the non-axisymmetric ablator of the heat shield is approximated by an axisymmetric ablator. Since the stiffness of the ablator is reduced at high temperatures, this approximation is justified at the temperature of re-entry. However, at low temperatures the stiffness of the ablator, as compared to the stiffness of the sub-structure, is significant.

Figure 4.8 illustrates typical stresses developed within two different heat shields when subjected to a uniform reduction in temperature (185°F to -250°F). The ablator thicknesses used in the analyses correspond to the thicknesses at sections 0° and 180° (Fig. 4.4). The resulting stresses are comparable which indicates that for an axisymmetric heat shield, the thickness of the ablator does not affect the stress distribution significantly. It is also reasonable to expect that the actual behavior of the non-axisymmetric heat shield will be bracketed by these results.

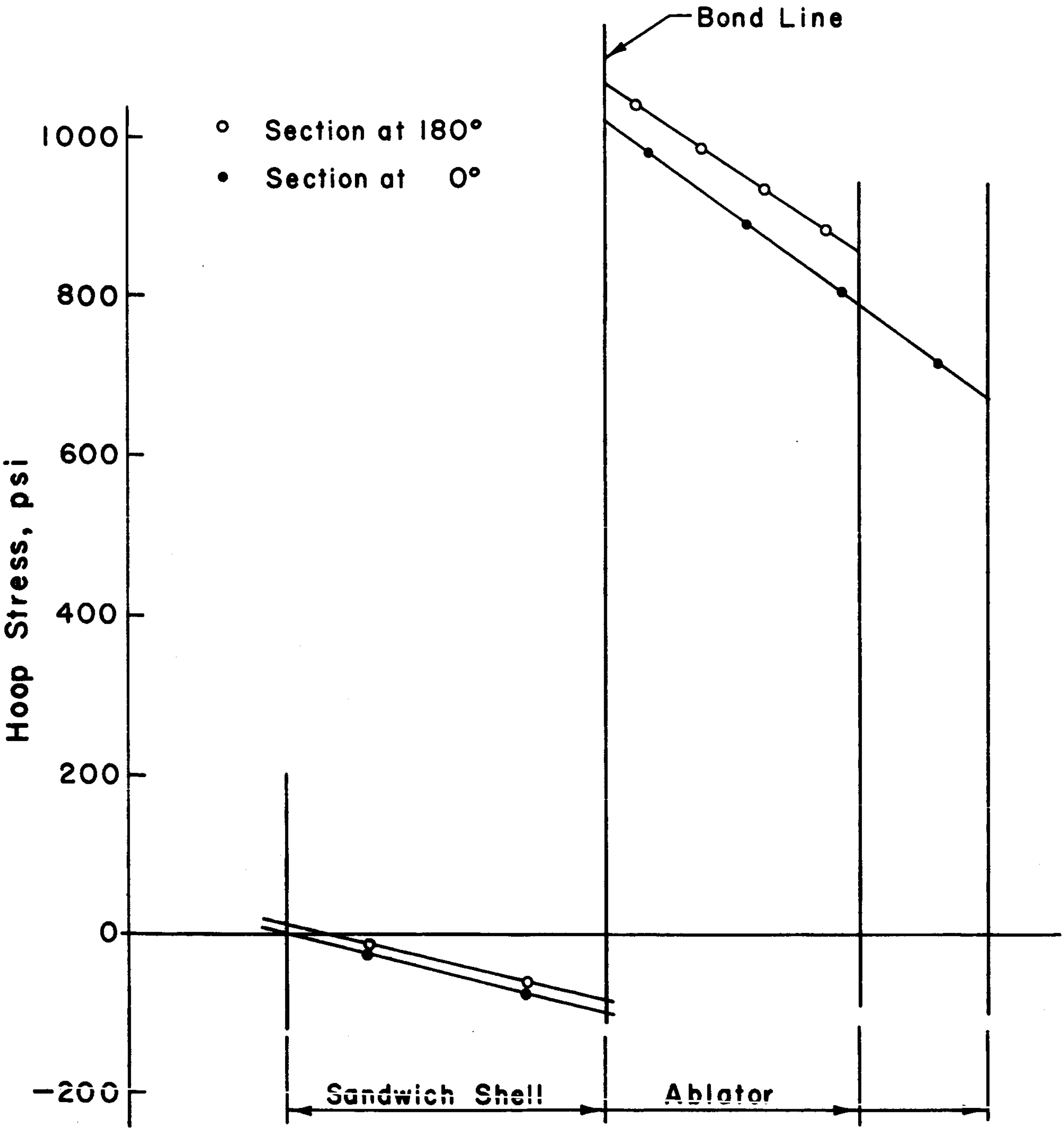


Fig. 4.8 Typical Results of Cold Soak Analysis

APPENDIX A

SOLUTION OF EQUILIBRIUM EQUATIONS

The equilibrium equations for a system of finite elements may be written in the following form:

$$A_{11} X_1 + A_{12} X_2 + A_{13} X_3 \text{ ----------------- } + A_{1N} X_N = B_1 \qquad \text{(A1a)}$$

$$A_{21} X_1 + A_{22} X_2 + A_{23} X_3 \text{ ----------------- } + A_{2N} X_N = B_2 \qquad \text{(A1b)}$$

$$A_{31} X_1 + A_{32} X_2 + A_{33} X_3 \text{ ----------------- } + A_{3N} X_N = B_3 \qquad \text{(A1c)}$$

$$A_{N1} X_1 + A_{N2} X_2 + A_{N3} X_3 \text{ ----------------- } + A_{NN} X_N = B_N \qquad (\text{ - })$$

or symbolically

$$\left[A\right]\left[X\right] = \left[B\right] \qquad \text{(A1)}$$

where $\left[A\right]$ = the stiffness matrix

$\left[X\right]$ = the unknown displacements

$\left[B\right]$ = the applied loads

Gaussian Elimination

The first step in the solution of the above set of equations is to solve Equation (A1a) for X_1, Or

$$X_1 = B_1/A_{11} - (A_{12}/A_{11}) X_2 - (A_{13}/A_{11}) X_3 \text{ -------------- } (A_{1N}/A_{11}) X_N \quad \text{(A2)}$$

If Equation (A2) is substituted into Equations (A1b, c, ..., N) a modified set of N-1 equations is determined.

$$A^1_{22} X_2 + A^1_{23} X_3 \text{ -------------- } + A^1_{2N} X_N = B^1_1 \quad \text{(A3a)}$$

$$A^1_{32} X_2 + A^1_{33} X_3 \text{ -------------- } + A^1_{3N} X_N = B^1_2 \quad \text{(A3b)}$$

$$A^1_{N2} X_2 + A^1_{N3} X_3 \text{ -------------- } + A^1_{NN} X_N = B^1_N$$

where $$A^1_{ij} = A_{ij} - A_{i1} A_{1j}/A_{11} \quad i, j = 2, ..., N \quad \text{(A4a)}$$

$$B^1_i = B_i - A_{i1} B_1/A_{11} \quad i = 2, ..., N \quad \text{(A4b)}$$

A similar procedure is used to eliminate X_2 from Equation (A3), etc. A general algorithm for the elimination of X_n may be written as

$$X_n = (B_n^{n-1}/A_{nn}^{n-1}) - \sum (A_{nj}^{n-1}/A_{nn}^{n-1})\, X_j \qquad j = n+1, \ldots, N \tag{A5}$$

$$A_{ij}^{n} = A_{ij}^{n-1} - A_{in}^{n-1}\,(A_{nj}^{n-1}/A_{nn}^{n-1}) \qquad i, j = n+1, \ldots, N \tag{A6}$$

$$B_i^{n} = B_i^{n-1} - A_{in}^{n-1}\,(B_n^{n-1}/A_{nn}^{n-1}) \qquad i = n+1, \ldots, N \tag{A7}$$

Equations A5, A6 and A7 may be rewritten in compact form:

$$X_n = D_n - \sum C_{nj} X_j \qquad j = n+1, \ldots, N \tag{A8}$$

$$A_{ij}^{n} = A_{ij}^{n-1} - A_{in}^{n-1}\, C_{nj} \qquad i, j = n+1, \ldots, N \tag{A9}$$

$$B_i^{n} = B_i^{n-1} - A_{in}^{n-1}\, D_n \qquad i = n+1, \ldots, N \tag{A10}$$

where
$$D_n = B_n^{n-1}/A_{nn}^{n-1}$$
$$C_{nj} = A_{nj}^{n-1}/A_{nn}^{n-1}$$

After the above procedure is applied N-1 times the original set of equations is reduced to the following single equation

$$A_{NN}^{N-1} X_N = B_N^{N-1}$$

which is solved directly for X_N

$$X_N = B_N^{N-1}/A_{NN}^{N-1}$$

In terms of the previous notation, this is

$$X_N = D_N \tag{A11}$$

The remaining unknowns are determined in reverse order by the repeated application of Equation (A8).

Simplification for Band Matrices

For many finite element systems it is possible to place the stiffness matrix in a "band" form which results in the concentration of the elements of the stiffness matrix along the main diagonal. Therefore, the following simplifications in the general algorithm (Equations A8, A9 and A10) are possible:

$$X_n = D_n - \sum C_{nj} X_j \qquad j = n+1, \ldots, n+M-1 \tag{A12}$$

$$A_{ij}^n = A_{ij}^{n-1} - A_{in}^{n-1} C_{nj} \qquad i, j = n+1, \ldots, n+M-1 \tag{A13}$$

$$B_i^n = B_i^{n-1} - A_{in}^{n-1} D_n \qquad i = n+1, \ldots, n+M-1 \tag{A14}$$

where M is the band width of the matrix.

The number of numerical operations can further be reduced by recognizing that the reduced matrix at any stage of procedure is symmetric. Accordingly, Equation (A13) may be replaced by the following equation:

$$A^n_{ij} = A^{n-1}_{ij} - A^{n-1}_{in} C_{nj} \qquad \begin{array}{l} i = n+1, \ldots, n+M-1 \\ j = i, \ldots, n+M-1 \end{array} \qquad (A15)$$

since

$$A^n_{ji} = A^n_{ij}$$

The number of numerical operations required for the solution of a band matrix is proportional to NM^2 as compared to N^3 which is required for the solution of a full matrix. Also, the computer storage required by the band matrix procedure is NM as compared to N^2 required by a set of N arbitrary equations.

This technique has been used in the automated axisymmetric program for the analysis of a typical heat shield idealized by a 10 x 40 mesh of quadrilateral elements. A solution to 800 simultaneous equations was necessary, which required less than two minutes of computing time on the IBM 7094.

APPENDIX B

MATRIX FORMULATION OF THE LEAST SQUARE CURVE-FIT PROCEDURE

Consider the problem of selecting the "best" polynomial of the form $y = C_1 + C_2X + C_3X^2 + C_4X^3, \ldots, C_NX^{N-1}$ to represent the following set of data points:

$$X_1, Y_1; X_2; Y_2; \text{---------------------}; X_M, Y_M$$

If the above polynomial is evaluated at points X_1 to X_M, M equation of the following form are found:

$$C_1 + C_2 X_m + C_2 X_m^2 + \text{-------------} C_N X_m^{N-1} = Y_m \quad m = 1, \ldots, M$$

or in matrix form

$$\begin{bmatrix} 1 & X_1 & X_1^2 & \text{-----} & X_1^{N-1} \\ 1 & X_2 & X_2^2 & \text{-----} & X_2^{N-1} \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ 1 & X_m & X_m^2 & \text{------} & X_m^{N-1} \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ 1 & X_M & X_M^2 & \text{------} & X_M^{N-1} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ - \\ - \\ - \\ C_N \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ \text{---} \\ Y_m \\ \text{---} \\ Y_M \end{bmatrix} \qquad \text{(B1a)}$$

or symbolically

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} C \end{bmatrix} = \begin{bmatrix} Y \end{bmatrix} \tag{B1b}$$

where

$\begin{bmatrix} A \end{bmatrix}$ = a M x N matrix

$\begin{bmatrix} C \end{bmatrix}$ = a N x 1 matrix

$\begin{bmatrix} Y \end{bmatrix}$ = a M x 1 matrix

If Equation (B1) is premultiplied by $\begin{bmatrix} A \end{bmatrix}^T$, a set of N linear equations in N unknowns is created. Consequently,

$$\begin{bmatrix} B \end{bmatrix} \begin{bmatrix} C \end{bmatrix} = \begin{bmatrix} D \end{bmatrix} \tag{B2}$$

where

$\begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} A \end{bmatrix}^T \begin{bmatrix} A \end{bmatrix}$ = a N x N matrix

$\begin{bmatrix} D \end{bmatrix} = \begin{bmatrix} A \end{bmatrix}^T \begin{bmatrix} Y \end{bmatrix}$ = a N x 1 matrix

Equation (B2) can now be solved directly for the unknown coefficients $\begin{bmatrix} C \end{bmatrix}$.

This procedure is numerically equivalent to the standard least square procedure. However, it is presented here in a form which is readily programmed for the digital computer The technique is not restricted to polynomials.

Figure B1 illustrates the application of the method in the evaluation of the elastic modulus for temperature dependent materials. A fourth order polynomial was used.

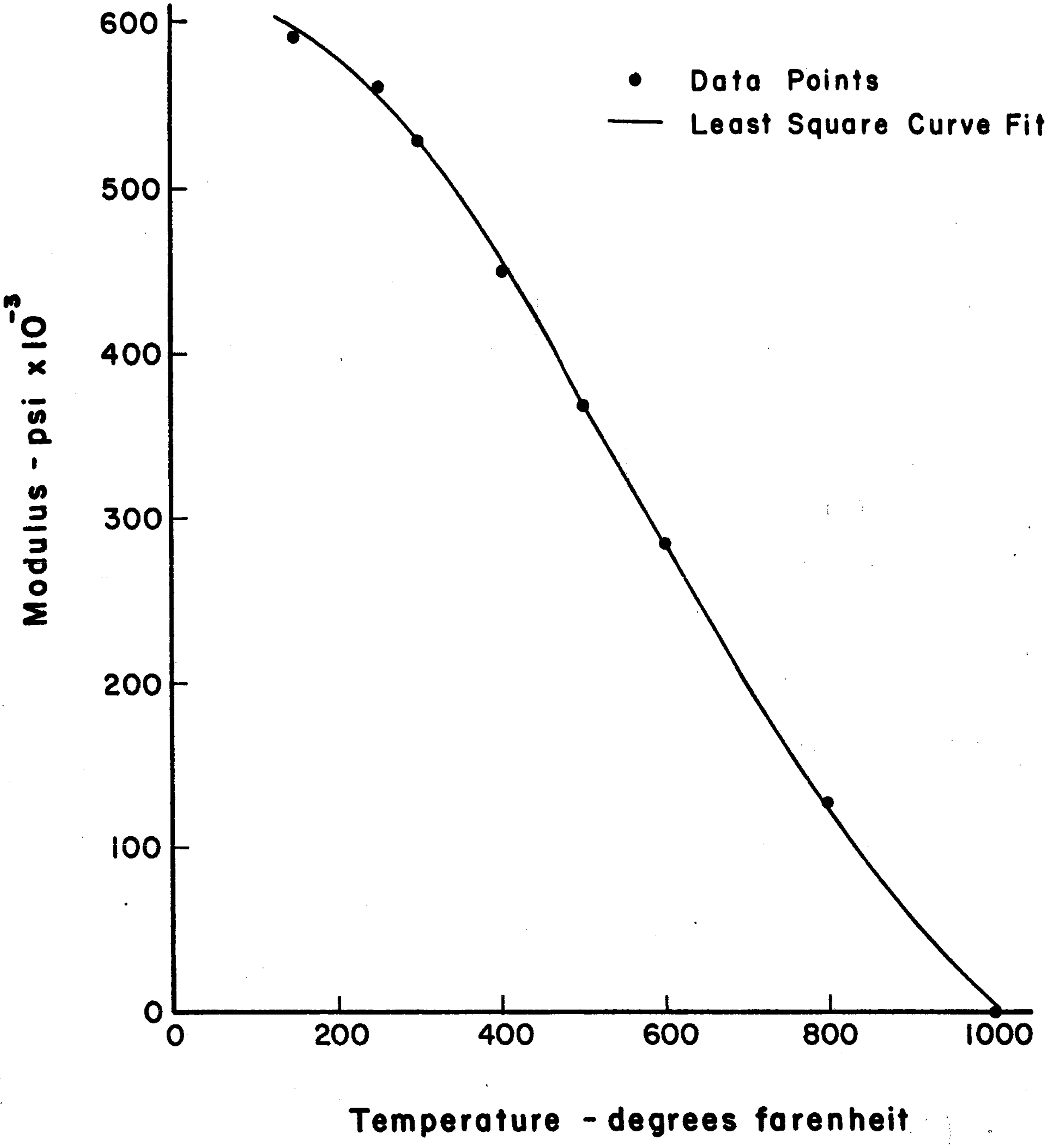


Fig. B1 Example of Least Square Curve Fit

APPENDIX C - MATHEMATICAL MODEL OF SANDWICH SHELL

The sandwich shell substructure is composed of a honeycomb core material and two steel face plates. The orthotropic core material is readily represented by solid triangular rings as indicated in Part IV of this report. However, for the description of the behavior of the face plates, a shell type element is used. The appropriate theory is given in this appendix.

It is assumed that the face plates are idealized by series of truncated cone elements which are connected at mesh points of the finite element system. The cross section of a typical truncated cone element is shown in Figure C1.

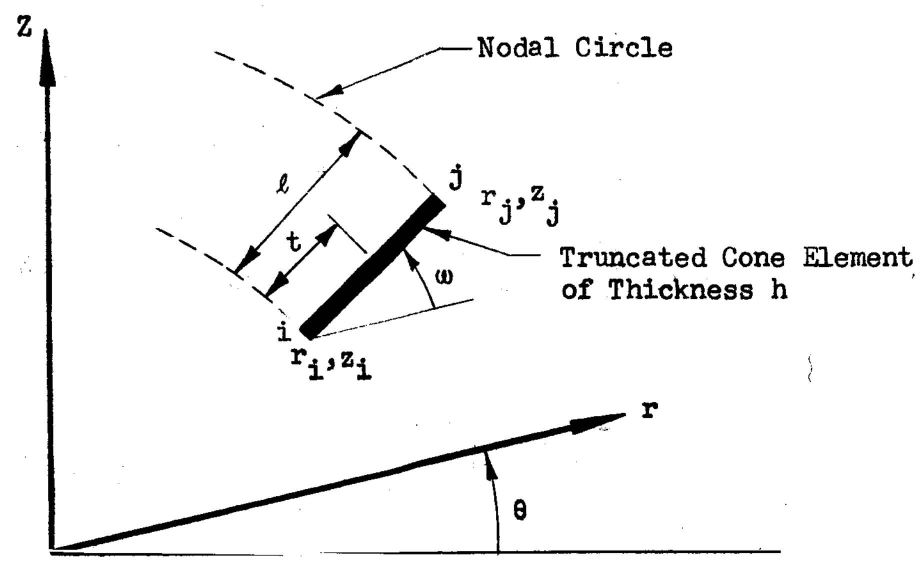


FIGURE C1 - CROSS SECTION OF TRUNCATED CONE ELEMENT

The displacements of the system in the r, θ, z coordinate system are written in the following form:

$$u_r = \sum u_{rn} (r,z) \cos n\theta$$

$$u_z = \sum u_{zn} (r,z) \cos n\theta$$

$$u_\theta = \sum u_{\theta n} (r,z) \sin n\theta$$

where $u_{rn} (r,z)$, $u_{zn} (r,z)$ and $u_{\theta n}(r,z)$ are the two-dimensional displacement functions (Fourier coefficients) associated with the harmonic n.

Within each truncated cone element, the two-dimensional displacement functions are assumed to vary linearly. The displacement at some point t within the element is given in terms of the nodal circle displacements by

$$u_r(t,\theta) = \sum \left[\frac{\ell-t}{\ell} u^i_{rn} + \frac{t}{\ell} u^j_{rn} \right] \cos n\theta \qquad \text{(C1)}$$

$$u_z(t,\theta) = \sum \left[\frac{\ell-t}{\ell} u^i_{zn} + \frac{t}{\ell} u^j_{zn} \right] \cos n\theta \qquad \text{(C2)}$$

$$u_\theta(t,\theta) = \sum \left[\frac{\ell-t}{\ell} u^i_{\theta n} + \frac{t}{\ell} u^j_{\theta n} \right] \sin n\theta \qquad \text{(C3)}$$

Accordingly, the displacement in the t-direction is

$$u_t(t,\theta) = u_r(t,\theta)\cos\omega + u_z(t,\theta)\sin\omega \tag{C4}$$

where $\cos\omega = \frac{a}{\ell}$

$\sin\omega = \frac{b}{\ell}$

The inplane strains within the truncated cone element are

$$\epsilon_t = \frac{\partial u_t}{\partial t} = \sum \epsilon_{tn}\cos n\theta \tag{C5}$$

$$\epsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} = \sum \epsilon_{\theta n}\cos n\theta \tag{C6}$$

$$\gamma_{\theta t} = \frac{1}{r}\frac{\partial u_t}{\partial\theta} + \frac{\partial u_\theta}{\partial t} = \sum \gamma_{\theta tn}\sin n\theta \tag{C7}$$

By combining Equations (C1) through (C7), the inplane strains for harmonic n are expressed in terms of nodal circle displacements by the following matrix equations:

$$\begin{bmatrix} \epsilon_{tn} \\ \epsilon_{\theta n} \\ \gamma_{\theta tn} \end{bmatrix} = \begin{bmatrix} -\frac{a}{\ell^2} & -\frac{b}{\ell^2} & 0 & \frac{a}{\ell^2} & \frac{b}{\ell^2} & 0 \\ \frac{\ell-t}{r\ell} & 0 & \frac{n(\ell-t)}{r\ell} & \frac{t}{r\ell} & 0 & \frac{nt}{r\ell} \\ -\frac{na}{r\ell^2}(\ell-t) & -\frac{nb}{r\ell^2}(\ell-t) & -\frac{1}{\ell} & -\frac{nat}{r\ell^2} & -\frac{nbt}{r\ell^2} & \frac{1}{\ell} \end{bmatrix} \begin{bmatrix} u^i_{rn} \\ u^i_{zn} \\ u^i_{\theta n} \\ u^j_{rn} \\ u^j_{tn} \\ u^j_{\theta n} \end{bmatrix} \tag{C8}$$

If Equation (C8) is evaluated at the center of the element, the average element strains are given by

$$\begin{bmatrix} \bar{\epsilon}_{tn} \\ \bar{\epsilon}_{\theta n} \\ \bar{\gamma}_{\theta tn} \end{bmatrix} = \begin{bmatrix} -\frac{a}{\ell^2} & -\frac{b}{\ell^2} & 0 & \frac{a}{\ell^2} & \frac{b}{\ell^2} & 0 \\ \frac{1}{2\bar{r}} & 0 & \frac{n}{2\bar{r}} & \frac{1}{2\bar{r}} & 0 & \frac{n}{2\bar{r}} \\ -\frac{na}{2\bar{r}} & -\frac{nb}{2\bar{r}\ell} & -\frac{1}{\ell} & -\frac{na}{2\bar{r}\ell} & -\frac{nb}{2\bar{r}\ell} & \frac{1}{\ell} \end{bmatrix} \begin{bmatrix} u^i_{rn} \\ u^i_{zn} \\ u^i_{\theta n} \\ u^j_{rn} \\ u^j_{zn} \\ u^j_{\theta n} \end{bmatrix} \qquad \text{(C9a)}$$

where $\bar{r} = (r_i + r_j)/2$

or Equation (C9a) written in symbolic form

$$\left[\epsilon_n\right] = \left[G_n\right]\left[u_n\right] \qquad \text{(C9b)}$$

From Hooke's law, the stresses within the element are given by

$$\begin{bmatrix} \sigma_{tn} \\ \sigma_{\theta n} \\ \tau_{\theta tn} \end{bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{tn} \\ \epsilon_{\theta n} \\ \nu_{\theta tn} \end{bmatrix} - \begin{bmatrix} \tau_1 \\ \tau_2 \\ 0 \end{bmatrix} \qquad \text{(10a)}$$

where $\tau_1 = \tau_2 = \frac{1 + \nu}{1 - \nu^2} E\alpha_t T_n$

Equation (10a) expressed in symbolic form is

$$\left[\sigma_n\right] = \left[C\right]\left[\epsilon_n\right] + \left[\tau\right] \tag{10b}$$

From Equation (1.15), the nodal circle forces in terms of the nodal circle displacement for the harmonic n are given by

$$\left[S_n\right] = \left[k_n\right]\left[U_n\right] + \left[L_n\right] T_n \tag{C11}$$

where

$$\left[K_n\right] = \bar{r}A \left[G_n\right]^T \left[C\right]\left[G_n\right]$$

$$\left[L_n\right] = \bar{r}A \left[G_n\right]^T \left[\tau\right]$$

For the truncated cone element

$$A = h \cdot \ell$$

These forces are included into the overall equilibrium of the system for the harmonic n by the same technique used for the system of triangular rings. When n equals zero, these equations reduce to the axisymmetric case.

APPENDIX D

PROGRAM LISTING - ARBITRARY AXISYMMETRIC STRUCTURES

Report No. 5654-02 FS

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CAXISY    STRESS ANALYSIS OF AXISYMMETRIC SOLIDS                          AXISY001
C                                                                         AXISY002
C      DIMENSION AND COMMON STATEMENTS                                    AXISY003
C                                                                         AXISY004
       DIMENSION THDIS(6),THFRC(6)                                        AXISY005
       DIMENSION NPNUM(340),XORD(340),YORD(340),                          AXISY006
      1DSX(340),DSY(340),XLOAD(340),YLOAD(340),NP(340,10),SXX(340,9),     AXISY007
      2SXY(340,9),SYX(340,9),SYY(340,9),NAP(340)                          AXISY008
       DIMENSION NUME(550),NPI(550),NPJ(550),NPK(550),ET(550),XU(550),    AXISY009
      1RO(550),COED(550),DT(550),THERM(550),AJ(550),BJ(550),AK(550),      AXISY010
      2BK(550),SIGXX(550),SIGYY(550),SIGXY(550),SLOPE(340)                AXISY011
       DIMENSION NPB(340),NFIX(340),LM(3),A(6,6),B(6,6),S(6,6)            AXISY012
       COMMON SXX,SXY,SYX,SYY                                             AXISY013
       EQUIVALENCE (SIGXX,RO,NPB), (SIGYY,COED,NFIX), (SIGXY,DT,SLOPE)    AXISY014
C                                                                         AXISY015
C      READ AND PRINT OF DATA                                             AXISY016
C                                                                         AXISY017
       KINN=5                                                             AXISY018
       KOUT=6                                                             AXISY019
  150  READ INPUT TAPE KINN,100                                           AXISY020
       WRITE OUTPUT TAPE KOUT,99                                          AXISY021
       WRITE OUTPUT TAPE KOUT,100                                         AXISY022
       READ INPUT TAPE KINN,1,NUMEL,NUMNP,NUMBC,NCPIN,NOPIN,NCYCM,TOLER,  AXISY023
      1XFAC,T1                                                            AXISY024
       WRITE OUTPUT TAPE KOUT,101,NUMEL                                   AXISY025
       WRITE OUTPUT TAPE KOUT,102,NUMNP                                   AXISY026
       WRITE OUTPUT TAPE KOUT,103,NUMBC                                   AXISY027
       WRITE OUTPUT TAPE KOUT,104,NCPIN                                   AXISY028
       WRITE OUTPUT TAPE KOUT,105,NOPIN                                   AXISY029
       WRITE OUTPUT TAPE KOUT,106,NCYCM                                   AXISY030
       WRITE OUTPUT TAPE KOUT,107,TOLER                                   AXISY031
       WRITE OUTPUT TAPE KOUT,108,XFAC                                    AXISY032
       READ INPUT TAPE KINN,2,(NUME(N),NPI(N),NPJ(N),NPK(N),ET(N),RO(N),  AXISY033
      1XU(N),COED(N),DT(N),N=1,NUMEL)                                     AXISY034
       READ INPUT TAPE KINN,3,(NPNUM(M),XORD(M),YORD(M),XLOAD(M),YLOAD(M)AXISY035
      1,DSX(M),DSY(M),M=1,NUMNP)                                          AXISY036
       IF (T1) 160,155,160                                                AXISY037
  155  WRITE OUTPUT TAPE KOUT,110                                         AXISY038
       WRITE OUTPUT TAPE KOUT,2,(NUME(N),NPI(N),NPJ(N),NPK(N),ET(N),RO(N)AXISY039
      1,XU(N),COED(N),DT(N),N=1,NUMEL)                                    AXISY040
       WRITE OUTPUT TAPE KOUT,111                                         AXISY041
       WRITE OUTPUT TAPE KOUT,109,(NPNUM(M),XORD(M),YORD(M),XLOAD(M),     AXISY042
      1YLOAD(M),DSX(M),DSY(M),M=1,NUMNP)                                  AXISY043
C                                                                         AXISY044
```

```
C      INITIALIZATION
C
  160 NCYCLE=0
      NUMPT=NCPIN
      NUMOPT=NOPIN
      DO 175 L=1,NUMNP
      DO 170 M=1,9
      SXX(L,M)=0.0
      SXY(L,M)=0.0
      SYX(L,M)=0.0
      SYY(L,M)=0.0
  170 NP(L,M)=0
      NP(L,10)=0
  175 NP(L,1)=L
C
C      FOR EACH ELEMENT OF THE SYSTEM
C
      NTAG=0
      DO 2000 N=1,NUMEL
C
C       1. INITIALIZATION
C
      I=NPI(N)
      J=NPJ(N)
      K=NPK(N)
      THICK=(XORD(I)+XORD(J)+XORD(K))/3.
      AJ(N)=XORD(J)-XORD(I)
      AK(N)=XORD(K)-XORD(I)
      BJ(N)=YORD(J)-YORD(I)
      BK(N)=YORD(K)-YORD(I)
  176 AREA=(AJ(N)*BK(N)-BJ(N)*AK(N))/2.
      IF (AREA) 701,701,177
  701 WRITE OUTPUT TAPE KOUT,711,(N)
      NTAG=1
  177 COMM=(1.+XU(N))*(1.-2.*XU(N))
      THERM(N)=-ET(N)*COED(N)*DT(N)/(1.-2.XU(N))
C
C       2. COMPUTE ELEMENT STIFFNESS MATRIX
C
C         A.  FORM MATRIX (A)
C
      A(1,1)=BJ(N)-BK(N)
      A(1,2)=0.0
      A(1,3)=BK(N)
```

```
      A(1,4)=0.0                                                        AXISY089
      A(1,5)=-BJ(N)                                                     AXISY090
      A(1,6)=0.0                                                        AXISY091
      A(2,1)=0.0                                                        AXISY092
      A(2,2)=AK(N)-AJ(N)                                                AXISY093
      A(2,3)=0.0                                                        AXISY094
      A(2,4)=-AK(N)                                                     AXISY095
      A(2,5)=0.0                                                        AXISY096
      A(2,6)=AJ(N)                                                      AXISY097
      A(3,1)=AK(N)-AJ(N)                                                AXISY098
      A(3,2)=BJ(N)-BK(N)                                                AXISY099
      A(3,3)=-AK(N)                                                     AXISY100
      A(3,4)=BK(N)                                                      AXISY101
      A(3,5)=AJ(N)                                                      AXISY102
      A(3,6)=-BJ(N)                                                     AXISY103
      C=.66666667*AREA                                                  AXISY104
      A(4,1)=C/XORD(I)                                                  AXISY105
      A(4,2)=0.0                                                        AXISY106
      A(4,3)=C/XORD(J)                                                  AXISY107
      A(4,4)=0.0                                                        AXISY108
      A(4,5)=C/XORD(K)                                                  AXISY109
      A(4,6)=0.0                                                        AXISY110
C                                                                       AXISY111
C        B.  FORM MATRIX (B)                                            AXISY112
C                                                                       AXISY113
      COMM=.25*ET(N)*THICK/(COMM*AREA)                                  AXISY114
      E=XU(N)*COMM                                                      AXISY115
      D=(1.-XU(N))*COMM                                                 AXISY116
      C=(.5-XU(N))*COMM                                                 AXISY117
      B(1,1)=D                                                          AXISY118
      B(1,2)=E                                                          AXISY119
      B(1,3)=0.0                                                        AXISY120
      B(1,4)=E                                                          AXISY121
      B(2,1)=E                                                          AXISY122
      B(2,2)=D                                                          AXISY123
      B(2,3)=0.0                                                        AXISY124
      B(2,4)=E                                                          AXISY125
      B(3,1)=0.0                                                        AXISY126
      B(3,2)=0.0                                                        AXISY127
      B(3,3)=C                                                          AXISY128
      B(3,4)=0.0                                                        AXISY129
      B(4,1)=E                                                          AXISY130
      B(4,2)=E                                                          AXISY131
      B(4,3)=0.0                                                        AXISY132
```

```
      B(4,4)=D
C
C        C.  FORM ELEMENT STIFFNESS MATRIX (A)T*(B)*(A)
C
      DO 182 J=1,6
      DO 182 I=1,4
      S(I,J)=0.0
      DO 182 K=1,4
  182 S(I,J)=S(I,J)+B(I,K)*A(K,J)
      DO 183 J=1,6
      DO 183 I=1,4
  183 B(J,I)=S(I,J)
      DO 184 J=1,6
      DO 184 I=1,6
      S(I,J)=0.0
      DO 184 K=1,4
  184 S(I,J)=S(I,J)+B(I,K)*A(K,J)
C
C       3. ADD ELEMENT STIFFNESS TO TOTAL STIFFNESS
C
      LM(1)=NPI(N)
      LM(2)=NPJ(N)
      LM(3)=NPK(N)
      DO 200 L=1,3
      DO 200 M=1,3
      LX=LM(L)
      MX=0
  185 MX=MX+1
      IF(NP(LX,MX)-LM(M)) 190,195,190
  190 IF(NP(LX,MX)) 185,195,185
  195 NP(LX,MX)=LM(M)
      IF (MX-10) 196,702,702
  702 WRITE OUTPUT TAPE KOUT,712,(LX)
      NTAG=1
  196 SXX(LX,MX)=SXX(LX,MX)+S(2*L-1,2*M-1)
      SXY(LX,MX)=SXY(LX,MX)+S(2*L-1,2*M)
      SYX(LX,MX)=SYX(LX,MX)+S(2*L,2*M-1)
  200 SYY(LX,MX)=SYY(LX,MX)+S(2*L,2*M)
C
C       5. COMPUTE BODY FORCES
C
      DL=AREA*THICK*RO(N)/3.
      I=NPI(N)
      J=NPJ(N)
```

```
      K=NPK(N)                                                          AXISY177
      C=COED(N)*DT(N)                                                   AXISY178
      THDIS(1)=C*XORD(I)                                                AXISY179
      THDIS(2)=0.0                                                      AXISY180
      THDIS(3)=C*XORD(J)                                                AXISY181
      THDIS(4)=C*BJ(N)                                                  AXISY182
      THDIS(5)=C*XORD(K)                                                AXISY183
      THDIS(6)=C*BK(N)                                                  AXISY184
      DO 204 L=1,6                                                      AXISY185
      THFRC(L)=0.0                                                      AXISY186
      DO 204 M=1,6                                                      AXISY187
  204 THFRC(L)=THFRC(L)+S(L,M)*THDIS(M)                                 AXISY188
      XLOAD(I)=XLOAD(I)+THFRC(1)                                        AXISY189
      YLOAD(I)=YLOAD(I)+THFRC(2)-DL                                     AXISY190
      XLOAD(J)=XLOAD(J)+THFRC(3)                                        AXISY191
      YLOAD(J)=YLOAD(J)+THFRC(4)-DL                                     AXISY192
      XLOAD(K)=XLOAD(K)+THFRC(5)                                        AXISY193
      YLOAD(K)=YLOAD(K)+THFRC(6)-DL                                     AXISY194
 2000 CONTINUE                                                          AXISY195
C                                                                       AXISY196
C     COUNT OF ADJACENT NODAL POINTS                                    AXISY197
C                                                                       AXISY198
      DO 206 M=1,NUMNP                                                  AXISY199
      MX =1                                                             AXISY200
  205 MX=MX+1                                                           AXISY201
      IF (NP(M,MX)) 206,206,205                                         AXISY202
  206 NAP(M)=MX-1                                                       AXISY203
C                                                                       AXISY204
C     INVERSION OF NODAL POINT STIFFNESS                                AXISY205
C                                                                       AXISY206
      DO 210 M=1,NUMNP                                                  AXISY207
      COMM=SXX(M,1)*SYY(M,1)-SXY(M,1)*SYX(M,1)                          AXISY208
      TEMP=SYY(M,1)/COMM                                                AXISY209
      SYY(M,1)=SXX(M,1)/COMM                                            AXISY210
      SXX(M,1)=TEMP                                                     AXISY211
      SXY(M,1)=-SXY(M,1)/COMM                                           AXISY212
  210 SYX(M,1)=-SYX(M,1)/COMM                                           AXISY213
C                                                                       AXISY214
C     MODIFICATION OF BOUNDARY FLEXIBILITIES                            AXISY215
C                                                                       AXISY216
      WRITE OUTPUT TAPE KOUT,112                                        AXISY217
      READ INPUT TAPE KINN,4,(NPB(L),NFIX(L),SLOPE(L),L=1,NUMBC)        AXISY218
      WRITE OUTPUT TAPE KOUT,4,(NPB(L),NFIX(L),SLOPE(L),L=1,NUMBC)      AXISY219
      DO 240 L=1,NUMBC                                                  AXISY220
```

```
      M=NPB(L)                                                          AXISY221
      NP(M,1)=0                                                         AXISY222
      IF(NFIX(L)-1) 225,220,215                                         AXISY223
  215 C=(SXX(M,1)*SLOPE(L)-SXY(M,1))/(SYX(M,1)*SLOPE(L)-SYY(M,1))       AXISY224
      R=1.-C*SLOPE(L)                                                   AXISY225
      SXX(M,1)=(SXX(M,1)-C*SYX(M,1))/R                                  AXISY226
      SXY(M,1)=(SXY(M,1)-C*SYY(M,1))/R                                  AXISY227
      SYX(M,1)=SXX(M,1)*SLOPE(L)                                        AXISY228
       SYY(M,1)=SXY(M,1)*SLOPE(L)                                       AXISY229
      GO TO 240                                                         AXISY230
  220 SYY(M,1)=SYY(M,1)-SYX(M,1)*SXY(M,1)/SXX(M,1)                      AXISY231
      GO TO 230                                                         AXISY232
  225 SYY(M,1)=0.0                                                      AXISY233
  230 SXX(M,1)=0.0                                                      AXISY234
  235 SXY(M,1)=0.0                                                      AXISY235
      SYX(M,1)=0.0                                                      AXISY236
  240 CONTINUE                                                          AXISY237
C                                                                       AXISY238
C      ITERATION ON NODAL POINT DISPLACEMENTS                           AXISY239
C                                                                       AXISY240
      IF (NTAG) 150,243,150                                             AXISY241
  243 WRITE OUTPUT TAPE KOUT,119                                        AXISY242
  244 SUM=0.0                                                           AXISY243
      DO 290 M=1,NUMNP                                                  AXISY244
      NUM=NAP(M)                                                        AXISY245
      IF (SXX(M,1)+SYY(M,1)) 275,290,275                                AXISY246
  275 FRX=XLOAD(M)                                                      AXISY247
      FRY=YLOAD(M)                                                      AXISY248
      DO 280 L=2,NUM                                                    AXISY249
      N=NP(M,L)                                                         AXISY250
      FRX=FRX-SXX(M,L)*DSX(N)-SXY(M,L)*DSY(N)                           AXISY251
  280 FRY=FRY-SYX(M,L)*DSX(N)-SYY(M,L)*DSY(N)                           AXISY252
      DX=SXX(M,1)*FRX+SXY(M,1)*FRY-DSX(M)                               AXISY253
      DY=SYX(M,1)*FRX+SYY(M,1)*FRY-DSY(M)                               AXISY254
      DSX(M)=DSX(M)+XFAC*DX                                             AXISY255
      DSY(M)=DSY(M)+XFAC*DY                                             AXISY256
  285 SUM=SUM+(ABSF(DX)+ABSF(DY))/(SXX(M,1)+SYY(M,1))                   AXISY257
  290 CONTINUE                                                          AXISY258
C                                                                       AXISY259
C     CYCLE COUNT AND PRINT CHECK                                       AXISY260
C                                                                       AXISY261
      NCYCLE=NCYCLE +1                                                  AXISY262
      IF (NCYCLE-NUMPT) 305,300,300                                     AXISY263
  300 NUMPT=NUMPT+NCPIN                                                 AXISY264
```

```
      WRITE OUTPUT TAPE KOUT,120,NCYCLE,SUM                              AXISY265
  305 IF (SUM-TOLER) 400,400,310                                         AXISY266
  310 IF(NCYCM-NCYCLE) 400,400,315                                       AXISY267
  315 IF (NCYCLE-NUMOPT) 244,320,320                                     AXISY268
  320 NUMOPT=NUMOPT+NOPIN                                                AXISY269
C                                                                        AXISY270
C     PRINT OF NODAL POINT DISPLACEMENTS                                 AXISY271
C                                                                        AXISY272
  400 WRITE OUTPUT TAPE KOUT,99                                          AXISY273
      WRITE OUTPUT TAPE KOUT,100                                         AXISY274
      WRITE OUTPUT TAPE KOUT,121                                         AXISY275
      WRITE OUTPUT TAPE KOUT,122,(NPNUM(M),DSX(M),DSY(M),M=1,NUMNP)      AXISY276
C                                                                        AXISY277
C     CALCULATION AND PRINT OF ELEMENT STRESSES                          AXISY278
C                                                                        AXISY279
      WRITE OUTPUT TAPE KOUT,123                                         AXISY280
      DO 420 N=1,NUMEL                                                   AXISY281
      I=NPI(N)                                                           AXISY282
      J=NPJ(N)                                                           AXISY283
      K=NPK(N)                                                           AXISY284
C                                                                        AXISY285
C      1. COMPUTE ELEMENT STRAINS                                        AXISY286
C                                                                        AXISY287
      C=AJ(N)*BK(N)-BJ(N)*AK(N)                                          AXISY288
      EPX=((BJ(N)-BK(N))*DSX(I)+BK(N)*DSX(J)-BJ(N)*DSX(K))/C             AXISY289
      EPY=((AK(N)-AJ(N))*DSY(I)-AK(N)*DSY(J)+AJ(N)*DSY(K))/C             AXISY290
      EPZ=(DSX(I)+DSX(J)+DSX(K))/(XORD(I)+ XORD(J)+XORD(K))              AXISY291
      GAM=(  (AK(N)-AJ(N))*DSX(I)-AK(N)*DSX(J)+AJ(N)*DSX(K)              AXISY292
     1   +(BJ(N)-BK(N))*DSY(I)+BK(N)*DSY(J)-BJ(N)*DSY(K)  )/C            AXISY293
C                                                                        AXISY294
C      2. COMPUTE ELEMENT STRESSES                                       AXISY295
C                                                                        AXISY296
      E=ET(N)*XU(N)/((1.+XU(N))*(1.-2.*XU(N)))                           AXISY297
      D=E*(1.-XU(N))/XU(N)                                               AXISY298
      C=E*(1.-2.*XU(N))/(2.*XU(N))                                       AXISY299
      X= D*EPX + E*EPY + E*EPZ +THERM(N)                                 AXISY300
      Y= E*EPX + D*EPY + E*EPZ +THERM(N)                                 AXISY301
      Z= E*EPX + E*EPY + D*EPZ +THERM(N)                                 AXISY302
      XY=C*GAM                                                           AXISY303
      SIGXX(N)=X                                                         AXISY304
      SIGYY(N)=Y                                                         AXISY305
      SIGXY(N)=XY                                                        AXISY306
C                                                                        AXISY307
C      3.COMPUTE PRINCIPAL STRESSES                                      AXISY308
```

```
C
      C=(X+Y)/2.0
      R=SQRTF(((Y-X)/2.0)**2+XY**2)
      XMAX=C+R
      XMIN=C-R
      PA=0.5*57.29578*ATANF ( 2.* XY/(Y-X))
      IF (2.*X-XMAX-XMIN) 405,420,420
  405 IF (PA) 410,420,415
  410 PA=PA+90.0
      GO TO 420
  415 PA=PA-90.0
C
C      4.PRINT OF ELEMENT STRESSES
C
  420 WRITE OUTPUT TAPE KOUT,125,(NUME(N),X,Y,Z,XY,XMAX,XMIN,PA)
C
C     CALCULATION AND PRINT OF NODAL POINT STRESSES
C
      WRITE OUTPUT TAPE KOUT,823
      DO 900  M=1,NUMNP
       X=0.0
       Y=0.0
      XY=0.0
      SRX=0.0
      SRY=0.0
      E=0.0
      XX=0.0
      TEMP=0.0
      R=0.0
      DO 860 N=1,NUMEL
      I=NPI(N)
      J=NPJ(N)
      K=NPK(N)
      IF (M-I) 830,850,830
  830 IF (M-J) 835,845,835
  835 IF (M-K) 860,840,860
  840 I=NPK(N)
      K=NPI(N)
      GO TO 850
  845 I=NPJ(N)
      J=NPI(N)
  850 A=ABSF(XORD(J)+XORD(K)-2.*XORD(I))
      B=ABSF(YORD(J)+YORD(K)-2.*YORD(I))
      RY=B/(A+B)
```

```
      SRY=SRY+RY
      Y=Y+SIGYY(N)*RY
      RX=A/(A+B)
      SRX=SRX+RX
      X=X+SIGXX(N)*RX
      R=R+1.0
      XY=XY+SIGXY(N)
      E=E+ET(N)
      XX=XX+XU(N)
      TEMP=TEMP+THERM(N)*(1.-2.XU(N))
  860 CONTINUE
      X=X/SRX
      Y=Y/SRY
      XY=XY/R
      E=E/R
      XX=XX/R
      TEMP=TEMP/R
      Z=E*DSX(M)/XORD(M)+XX*(X+Y)+TEMP
      C=(X+Y)/2.0
      R=SQRTF(((Y-X)/2.0)**2+XY**2)
      XMAX=C+R
      XMIN=C-R
      PA=0.5*57.29578*ATANF ( 2.* XY/(Y-X))
      IF (2.*X-XMAX-XMIN) 805,820,820
  805 IF (PA) 810,820,815
  810 PA=PA+90.0
      GO TO 820
  815 PA=PA-90.0
  820 WRITE OUTPUT TAPE KOUT,125, (M,X,Y,Z,XY,XMAX,XMIN,PA)
  900 CONTINUE
C
      IF (SUM-TOLER) 440,440,430
  430 IF (NCYCM-NCYCLE) 440,440,243
C
  440 GO TO 150
C
C        FORMAT STATEMENTS
C
    1 FORMAT (6I4,2E12.5,4I1)
    2 FORMAT (4I4,4E12.4,F8.4)
    3 FORMAT (1I4,4F8.0,2F12.8)
    4 FORMAT (2I4,1F8.3)
    5 FORMAT (3E15.8)
   99 FORMAT (1H1)
```

```
100 FORMAT (72H  BCD INFORMATION                                          AXISY397
   1                )                                                     AXISY398
101 FORMAT(29HONUMBER OF ELEMENTS         =1I4/)                          AXISY399
102 FORMAT(29H NUMBER OF NODAL POINTS     =1I4/)                          AXISY400
103 FORMAT(29H NUMBER OF BOUNDARY POINTS  =1I4/)                          AXISY401
104 FORMAT(29H CYCLE PRINT INTERVAL       =1I4/)                          AXISY402
105 FORMAT(29H OUTPUT INTERVAL OF RESULTS =1I4/)                          AXISY403
106 FORMAT(29H CYCLE LIMIT                =1I4/)                          AXISY404
107 FORMAT(29H TOLERANCE LIMIT            =1E12.4/)                       AXISY405
108 FORMAT(29H OVER RELAXATION FACTOR     =1F6.3)                         AXISY406
109 FORMAT (1I8,4F12.1,2F12.8)                                            AXISY407
110 FORMAT (74H1EL.    I    J    K        E         DENSITY      POISSON  AXISY408
   1  ALPHA      DELTA T)                                                 AXISY409
111 FORMAT (80H1       NP         X-ORD         Y-ORD        X-LOAD       Y-LOAAXISY410
   1D        X-DISP        Y-DISP)                                        AXISY411
112 FORMAT (20H BOUNDARY CONDITIONS)                                      AXISY412
119 FORMAT(34H0            CYCLE       FORCE UNBALANCE)                   AXISY413
120 FORMAT (1I12,1E20.6)                                                  AXISY414
121 FORMAT (42HONODAL POINT X-DISPLACEMENT Y-DISPLACEMENT)                AXISY415
122 FORMAT (1I12,2E15.6)                                                  AXISY416
123 FORMAT(120H1  ELEMENT      X-STRESS         Y-STRESS          Z-STREAXISY417
   1SS          XY-STRESS       MAX-STRESS     MIN-STRESS       DIRECTION)AXISY418
124 FORMAT (1I10,3F20.4,5X,3F15.2)                                        AXISY419
125 FORMAT (1I6,4F17.3,1X,3F15.2)                                         AXISY420
711 FORMAT (32HOZERO OR NEGATIVE AREA, EL. NO.=1I4)                       AXISY421
712 FORMAT (33HOOVER 8 N.P. ADJACENT TO N.P. NO.1I4)                      AXISY422
823 FORMAT(120H1  N-POINT      X-STRESS         Y-STRESS          Z-STREAXISY423
   1SS          XY-STRESS       MAX-STRESS     MIN-STRESS       DIRECTION)AXISY424
C                                                                         AXISY425
    END                                                                   AXIS0426
```

APPENDIX E

PROGRAM LISTING - AXISYMMETRIC HEAT SHIELDS

```
CASHS         ANALYSIS OF AXISYMMETRIC HEAT SHIELDS
C
      DIMENSION
     1 XX(5),YY(5),S(10,10),P(10),X(40,10),Y(40,10),U(40,10),V(40,10),
     1 SS(800,24),RR(800),KODE(40,10),HED(12),T(40,10),CC(5,3)
     4 ,A(4,4),B(4,4)
C
      COMMON KINN,KOUT,NMAX,MMAX,MB,NMT,NUMLD,NUMBC,TE,TR,
     1 ES,XS,AS,TS,ESH,XSH,ASH,TSH,
     1 HED,X,Y,T,U,V,CC,XX,YY,S,P,KODE,RR,SS
C
C     READ AND PRINT OF DATA
C
   50 KINN=5
      KOUT=6
C
      CALL INPUT
C
      MX=2*MMAX
      NS=NMAX*MMAX
      MM=MMAX-1
      NN=NMAX-1
      MBAND=2*(MMAX+2)
      NEQ=2*NS
C
C     FORMATION OF STIFFNESS ARRAY
C
      DO 175 I=1,NEQ
      DO 170 J=1,MBAND
  170 SS(I,J)=0.0
  175 RR(I)=0.0
C
      DO 200 N=1,NN
      DO 200 M=1,MM
      DO 180  I=1,10
      P(I)=0.0
      DO 180  J=1,10
  180 S(I,J)=0.0
      XX(1)=X(N,M)
      XX(2)=X(N,M+1)
      XX(3)=X(N+1,M)
      XX(4)=X(N+1,M+1)
```

```
      YY(1)=Y(N,M)                                               ASHS0043
      YY(2)=Y(N,M+1)                                             ASHS0044
      YY(3)=Y(N+1,M)                                             ASHS0045
      YY(4)=Y(N+1,M+1)                                           ASHS0046
      NNN=MB-M                                                   ASHS0047
      TT=.25*(T(N,M)+T(N,M+1)+T(N+1,M)+T(N+1,M+1))               ASHS0048
      CALL TRIST(1,4,2,TT,NNN)                                   ASHS0049
      CALL TRIST(1,3,4,TT,NNN)                                   ASHS0050
C                                                                ASHS0051
      DO 200 I=1,4                                               ASHS0052
      II=((N-1)*MMAX+M-1)*2+I                                    ASHS0053
      KK=II+MX                                                   ASHS0054
      RR(II)=RR(II)+P(I)                                         ASHS0055
      RR(KK)=RR(KK)+P(I+4)                                       ASHS0056
      DO 190 J=I,4                                               ASHS0057
      JJ=J-I+1                                                   ASHS0058
      SS(II,JJ)=SS(II,JJ)+S(I,J)                                 ASHS0059
  190 SS(KK,JJ)=SS(KK,JJ)+S(I+4,J+4)                             ASHS0060
      DO 200 J=1,4                                               ASHS0061
      JJ=J-I+1+MX                                                ASHS0062
  200 SS(II,JJ)=SS(II,JJ)+S(I,J+4)                               ASHS0063
C                                                                ASHS0064
C     STIFFNESS OF PLATE ELEMENTS                                ASHS0065
C                                                                ASHS0066
      M=MB                                                       ASHS0067
      DO 500 N=1,NN                                              ASHS0068
C                                                                ASHS0069
      RI=X(N,M)                                                  ASHS0070
      ZI=Y(N,M)                                                  ASHS0071
      RJ=X(N+1,M)                                                ASHS0072
      ZJ=Y(N+1,M)                                                ASHS0073
      AA=ZJ-ZI                                                   ASHS0074
      BB=RJ-RI                                                   ASHS0075
      XL2=AA**2+BB**2                                            ASHS0076
      XL=SQRTF(XL2)                                              ASHS0077
C                                                                ASHS0078
      A(1,1)=1.0/(RI+RJ)                                         ASHS0079
      A(1,2)=0.0                                                 ASHS0080
      A(1,3)=A(1,1)                                              ASHS0081
      A(1,4)=0.0                                                 ASHS0082
      A(2,1)=-BB/XL2                                             ASHS0083
      A(2,2)=-AA/XL2                                             ASHS0084
```

```
      A(2,3)=-A(2,1)                                                    ASHS0085
      A(2,4)=-A(2,2)                                                    ASHS0086
C                                                                       ASHS0087
      B(1,1)=(ESH/(1.-XSH**2))*TSH*XL*(RI+RJ)/2.                        ASHS0088
      B(1,2)=B(1,1)*XSH                                                 ASHS0089
      B(2,1)=B(1,2)                                                     ASHS0090
      B(2,2)=B(1,1)                                                     ASHS0091
C                                                                       ASHS0092
      DO 470 I=1,2                                                      ASHS0093
      DO 470 J=1,4                                                      ASHS0094
      S(I,J)=0.0                                                        ASHS0095
      DO 470 K=1,2                                                      ASHS0096
  470 S(I,J)=S(I,J)+B(I,K)*A(K,J)                                       ASHS0097
C                                                                       ASHS0098
      DO 480 I=1,4                                                      ASHS0099
      DO 480 J=1,4                                                      ASHS0100
      B(I,J)=0.0                                                        ASHS0101
      DO 480 K=1,2                                                      ASHS0102
  480 B(I,J)=B(I,J)+A(K,I)*S(K,J)                                       ASHS0103
C                                                                       ASHS0104
      AT=ASH*(T(N,M)+T(N+1,M))/2.                                       ASHS0105
      A(1)=AT*RI                                                        ASHS0106
      A(2)=AT*ZI                                                        ASHS0107
      A(3)=AT*RJ                                                        ASHS0108
      A(4)=AT*ZJ                                                        ASHS0109
C                                                                       ASHS0110
      DO 485 I=1,4                                                      ASHS0111
      P(I)=0.0                                                          ASHS0112
      DO 485 K=1,4                                                      ASHS0113
  485 P(I)=P(I)+B(I,K)*A(K)                                             ASHS0114
C                                                                       ASHS0115
      DO 495 I=1,2                                                      ASHS0116
      II=((N-1)*MMAX+MB-1)*2+I                                          ASHS0117
      KK=II+MX                                                          ASHS0118
      RR(II)=RR(II)+P(I)                                                ASHS0119
      RR(KK)=RR(KK)+P(I+2)                                              ASHS0120
      DO 490 J=I,2                                                      ASHS0121
      JJ=J-I+1                                                          ASHS0122
      SS(II,JJ)=SS(II,JJ)+B(I,J)                                        ASHS0123
  490 SS(KK,JJ)=SS(KK,JJ)+B(I+2,J+2)                                    ASHS0124
      DO 495 J=1,2                                                      ASHS0125
      JJ=J-I+1+MX                                                       ASHS0126
```

```
  495 SS(II,JJ)=SS(II,JJ)+B(I,J+2)                                     ASHS0127
C                                                                      ASHS0128
  500 CONTINUE                                                         ASHS0129
C                                                                      ASHS0130
C         READ IN SPECIFIED RADIAL AND AXIAL LOADS                     ASHS0131
C                                                                      ASHS0132
      IF(NUMLD) 505, 202, 505                                          ASHS0133
C                                                                      ASHS0134
  505 WRITE OUTPUT TAPE KOUT, 2000                                     ASHS0135
C                                                                      ASHS0136
      DO 550 I=1,NUMLD                                                 ASHS0137
      READ INPUT TAPE KINN, 1001,N,M,PR,PZ                             ASHS0138
      WRITE OUTPUT TAPE KOUT, 2001, N,M,PR,PZ                          ASHS0139
      K = 2*( (N-1)*MMAX+M ) -1                                        ASHS0140
      RR(K) = RR(K) + PR                                               ASHS0141
  550 RR(K+1) = RR(K+1) + PZ                                           ASHS0142
C                                                                      ASHS0143
C         BOUNDARY CONDITIONS                                          ASHS0144
C                                                                      ASHS0145
  202 DO 60 N=1,NMAX                                                   ASHS0146
      DO 60 M=1,MMAX                                                   ASHS0147
   60 KODE(N,M)=0                                                      ASHS0148
C                                                                      ASHS0149
C          FIX EDGE                                                    ASHS0150
C                                                                      ASHS0151
      DO 205 M=1,MMAX                                                  ASHS0152
  205 KODE(1,M)=1                                                      ASHS0153
C                                                                      ASHS0154
C         READ IN ADDITIONAL BOUNDARY CONDITIONS                       ASHS0155
C                                                                      ASHS0156
      WRITE OUTPUT TAPE KOUT, 2002                                     ASHS0157
C                                                                      ASHS0158
      DO 208 I=1,NUMBC                                                 ASHS0159
      READ INPUT TAPE KINN,1000,NNN,MMM,KKK                            ASHS0160
      WRITE OUTPUT TAPE KOUT, 1000, NNN,MMM,KKK                        ASHS0161
  208 KODE(NNN,MMM)=KKK                                                ASHS0162
C                                                                      ASHS0163
      KK=0                                                             ASHS0164
      DO 250 N=1,NMAX                                                  ASHS0165
      DO 250 M=1,MMAX                                                  ASHS0166
      IF (KODE(N,M)) 210,250,210                                       ASHS0167
  210 LL=1                                                             ASHS0168
```

```
      LH=2                                                              ASHS0169
      IF (KODE(N,M)-2) 211,212,213                                      ASHS0170
  211 LH=1                                                              ASHS0171
      GO TO 213                                                         ASHS0172
  212 LL=2                                                              ASHS0173
  213 DO 245 L=LL,LH                                                    ASHS0174
      II=KK+L                                                           ASHS0175
      DO 240 J=1,MBAND                                                  ASHS0176
      SS(II,J)=0.0                                                      ASHS0177
      III=II+1-J                                                        ASHS0178
      IF (III) 240,240,230                                              ASHS0179
  230 SS(III,J)=0.0                                                     ASHS0180
  240 CONTINUE                                                          ASHS0181
      SS(II,1)=1.                                                       ASHS0182
  245 RR(II)=0.0                                                        ASHS0183
  250 KK=KK+2                                                           ASHS0184
C                                                                       ASHS0185
C                                                                       ASHS0186
C     SOLVE FOR DISPLACEMENTS                                           ASHS0187
C                                                                       ASHS0188
      CALL SYMSOL(SS,RR,NEQ,MBAND)                                      ASHS0189
C                                                                       ASHS0190
      K=1                                                               ASHS0191
      DO 400 N=1,NMAX                                                   ASHS0192
      DO 400 M=1,MMAX                                                   ASHS0193
      U(N,M)=RR(K)                                                      ASHS0194
      V(N,M)=RR(K+1)                                                    ASHS0195
  400 K=K+2                                                             ASHS0196
C                                                                       ASHS0197
      WRITE OUTPUT TAPE KOUT,2003                                       ASHS0198
      CALL PRINTM(U,NMAX,MMAX,40)                                       ASHS0199
      WRITE OUTPUT TAPE KOUT,2004                                       ASHS0200
      CALL PRINTM(V,NMAX,MMAX,40)                                       ASHS0201
C                                                                       ASHS0202
C     CALCULATE AND PRINT STRESSES                                      ASHS0203
C                                                                       ASHS0204
      CALL STRESS                                                       ASHS0205
C                                                                       ASHS0206
      GO TO 50                                                          ASHS0207
C                                                                       ASHS0208
C     FORMAT STATEMENTS                                                 ASHS0209
C                                                                       ASHS0210
```

```
 1000 FORMAT (3I5)                                                      ASHS0211
 1001 FORMAT (2I5, 2F10.3)                                              ASHS0212
 2000 FORMAT (46H1 POINTS WITH SPECIFIED RADIAL AND AXIAL LOADS //      ASHS0213
     113X 26HN    M    R-LOAD    Z-LOAD)                                ASHS0214
 2001 FORMAT (9X 2I5, 2F10.3)                                           ASHS0215
 2002 FORMAT (44H1 POINTS WITH ADDITIONAL BOUNDARY CONDITIONS //        ASHS0216
     116H    N    M  CODE)                                              ASHS0217
 2003 FORMAT (16H1R-DISPLACEMENTS)                                      ASHS0218
 2004 FORMAT (16H1Z-DISPLACEMENTS)                                      ASHS0219
      END                                                               ASHS0220
```

```
      SUBROUTINE INPUT                                                  ASHS0221
C                                                                       ASHS0222
C     COMMON AND DIMENSION STATEMENTS                                   ASHS0223
C                                                                       ASHS0224
      DIMENSION HED(12),R(40,10),Z(40,10),T(40,10),CC(5,3),TA(40),      ASHS0225
     1 TT(50),A(50,5),B(5,5),D(5,3),EE(50,3),U(40,10),V(40,10)          ASHS0226
     2,KR(5)                                                            ASHS0227
C                                                                       ASHS0228
      COMMON KINN,KOUT,NMAX,MMAX,MB,NMT,NUMLD,NUMBC,TE,TR,              ASHS0229
     1 ES,XS,AS,TS,ESH,XSH,ASH,TSH,                                     ASHS0230
     1 HED,R,Z,T,U,V,CC,TA,TT,A,B,D,EE                                  ASHS0231
C                                                                       ASHS0232
C     READ AND PRINT INPUT DATA                                         ASHS0233
C                                                                       ASHS0234
      READ INPUT TAPE KINN,1000,                                        ASHS0235
     1HED,NMAX,MMAX,MB,NMT,NUMLD,NUMBC,TE,TR,ES,XS,AS,TS,ESH,XSH,ASH,TSHASHS0236
      WRITE OUTPUT TAPE KOUT,2000,                                      ASHS0237
     1HED,NMAX,MMAX,MB,NMT,NUMLD,NUMBC,TE,TR                            ASHS0238
C                                                                       ASHS0239
C     CHECK THE SIZE OF NMAX, MMAX, NMT AND LOCATION OF MB.  CHECK FOR  ASHS0240
C     ADDITIONAL BOUNDARY CONDITIONS.  IF ANY INPUT ERRORS EXIT.        ASHS0241
C                                                                       ASHS0242
      DO 1 I=1,5                                                        ASHS0243
    1 KR(I) = 0                                                         ASHS0244
C                                                                       ASHS0245
      IF (40-NMAX) 5, 10, 10                                            ASHS0246
    5 KR(1) = 1                                                         ASHS0247
   10 IF (10-MMAX) 15, 20, 20                                           ASHS0248
   15 KR(2) = 2                                                         ASHS0249
   20 IF (MMAX-MB) 25, 25, 30                                           ASHS0250
   25 KR(3) = 3                                                         ASHS0251
   30 IF (50-NMT) 35, 40, 40                                            ASHS0252
   35 KR(4) = 4                                                         ASHS0253
   40 IF (NUMBC) 45, 45, 50                                             ASHS0254
   45 KR(5) = 6                                                         ASHS0255
C                                                                       ASHS0256
   50 DO 65 I=1,5                                                       ASHS0257
      IF(KR(I)) 60, 65, 60                                              ASHS0258
   60 WRITE OUTPUT TAPE KOUT, 2011, KR(I)                               ASHS0259
      KR = 1                                                            ASHS0260
   65 CONTINUE                                                          ASHS0261
C                                                                       ASHS0262
      IF(KR) 70, 75, 70                                                 ASHS0263
   70 CALL EXIT                                                         ASHS0264
C                                                                       ASHS0265
   75 WRITE OUTPUT TAPE KOUT, 2006                                      ASHS0266
```

```
      WRITE OUTPUT TAPE KOUT,2008, ES,XS,AS,TS                          ASHS0267
      WRITE OUTPUT TAPE KOUT,2007                                       ASHS0268
      WRITE OUTPUT TAPE KOUT,2008, ESH,XSH,ASH,TSH                      ASHS0269
      READ INPUT TAPE KINN,1001,                                        ASHS0270
     1 (R(N,MB),Z(N,MB),T(N,MB),TA(N),N=1,NMAX)                         ASHS0271
      WRITE OUTPUT TAPE KOUT,2001,                                      ASHS0272
     1 (R(N,MB),Z(N,MB),T(N,MB),TA(N),N=1,NMAX)                         ASHS0273
C                                                                       ASHS0274
C     LEAST SQUARE EVALUATION OF MATERIAL PROPERTIES                    ASHS0275
C                                                                       ASHS0276
      READ INPUT TAPE KINN,1002,                                        ASHS0277
     1 (TT(I),EE(I,1),EE(I,2),EE(I,3),I=1,NMT)                          ASHS0278
      WRITE OUTPUT TAPE KOUT, 2009                                      ASHS0279
      WRITE OUTPUT TAPE KOUT,2002,                                      ASHS0280
     1 (TT(I),EE(I,1),EE(I,2),EE(I,3),I=1,NMT)                          ASHS0281
C                                                                       ASHS0282
      DO 100 I=1,NMT                                                    ASHS0283
      A(I,1)=1.0                                                        ASHS0284
      A(I,2)=TT(I)                                                      ASHS0285
      A(I,3)=TT(I)*TT(I)                                                ASHS0286
      A(I,4)=A(I,3)*TT(I)                                               ASHS0287
  100 A(I,5)=A(I,4)*TT(I)                                               ASHS0288
C                                                                       ASHS0289
      CALL LEAST(A,EE,B,D,CC,NMT,5,3)                                   ASHS0290
C                                                                       ASHS0291
      WRITE OUTPUT TAPE KOUT, 2010                                      ASHS0292
      WRITE OUTPUT TAPE KOUT,2002,                                      ASHS0293
     1 (TT(I),EE(I,1),EE(I,2),EE(I,3),I=1,NMT)                          ASHS0294
C                                                                       ASHS0295
C     GENERATE MESH                                                     ASHS0296
C                                                                       ASHS0297
      MC = MB+1                                                         ASHS0298
      MBB = MB-1                                                        ASHS0299
      SHL = FLOATF(MBB)                                                 ASHS0300
      ABL = FLOATF(MMAX-MB)                                             ASHS0301
      DO 200 N=1,NMAX                                                   ASHS0302
C                                                                       ASHS0303
C     CHECK FOR END POINTS                                              ASHS0304
C                                                                       ASHS0305
      NL = N-1                                                          ASHS0306
       NH=N+1                                                           ASHS0307
      IF(NL)160,150,160                                                 ASHS0308
  150 SS = 0.                                                           ASHS0309
      CX = 1.                                                           ASHS0310
      GO TO 185                                                         ASHS0311
  160 IF(N-NMAX)180,170,180                                             ASHS0312
```

```
  170  NH=N
C
C      COMPUTE SIN AND COS AT EACH POINT ON BOND LINE
C
  180 XX = R(  NH,MB) - R(NL,MB)
      YY = Z( NH ,MB) - Z(NL,MB)
      ZZ = SQRTF(XX**2 + YY**2)
      SS = YY/ZZ
      CX = XX/ZZ
C
C      COMPUTE COORDINATES OF POINTS IN CONSTANT THICKNESS SHELL
C
  185 DO 190 M=1,MBB
      TT = TS*FLOATF(MB-M)/SHL
      T(N,M) = T(N,MB)
      R(N,M) = R(N,MB) + TT*SS
  190 Z(N,M) = Z(N,MB) - TT*CX
C      COMPUTE COORDINATES OF POINTS IN VARIABLE THICKNESS ABLATER
C
      DO 200 M=MC,MMAX
      TT = TA(N)*FLOATF(M-MB)/ABL
      T(N,M) = T(N,MB) + (TE - T(N,MB)) * (TT/TA(N))**2
      R(N,M) = R(N,MB) - TT*SS
  200 Z(N,M) = Z(N,MB) + TT*CX
C
C      DUMP OUT R-ORDINATE TABLE AND Z-ORDINATE TABLE
C
      WRITE OUTPUT TAPE KOUT,2003
      CALL PRINTM (R,NMAX,MMAX,40)
      WRITE OUTPUT TAPE KOUT,2004
      CALL PRINTM (Z,NMAX,MMAX,40)
      WRITE OUTPUT TAPE KOUT,2005
      CALL PRINTM (T,NMAX,MMAX,40)
      RETURN
C
C      FORMAT STATEMENTS
C
 1000 FORMAT (12A6/6I5,2F10.2/4F10.2/4F10.2)
 1001 FORMAT (4F10.2)
 1002 FORMAT (4F10.2)
 2000 FORMAT ( 1H1 12A6/
     1 33HONUMBER OF POINTS ALONG LENGTH--- I3/
     2 33H NUMBER OF POINTS THRU THICKNESS- I3/
     3 33H LOCATION OF BOND LINE----------- I3/
     4 33H NUMBER OF PROPERTY CARDS-------- I3/
     5 33H NUMBER OF POINTS WITH R,Z LOADS- I3/
```

```
    6 33H ADDITIONAL BOUNDARY CONDITIONS-- I3/                                ASHS0359
    7 33H SURFACE TEMPERATURE OF ABLATOR-- F6.0/                              ASHS0360
    8 33H ZERO STRESS TEMPERATURE--------- F6.0)                              ASHS0361
 2001 FORMAT (15H1    R-ORDINATE 5X 10HZ-ORDINATE 5X 10HBOND TEMP. 3X         ASHS0362
    1 17HABLATOR THICKNESS / (3F15.3,1F20.4))                                 ASHS0363
 2002 FORMAT (15H0  TEMPERATURE 5X 10HMODULUS A 5X 9HMODULUS B   1X           ASHS0364
    120H COEFF. OF EXPANSION /(3F15.0,1E20.5))                                ASHS0365
 2003 FORMAT (14H1  R-ORDINATES )                                             ASHS0366
 2004 FORMAT (14H1  Z-ORDINATES )                                             ASHS0367
 2005 FORMAT (14H1  TEMPERATURE )                                             ASHS0368
 2006 FORMAT (37H0PROPERTIES OF SANDWICH CORE MATERIAL )                      ASHS0369
 2007 FORMAT (30H0PROPERTIES OF SANDWICH PLATES )                             ASHS0370
 2008 FORMAT                                                                  ASHS0371
    1(28H  MODULUS OF ELASTICITY----- F10.0/                                  ASHS0372
    2 28H  POISSONS RATIO------------ F10.4/                                  ASHS0373
    3 28H  COEFFICIENT OF EXPANSION-- F10.8/                                  ASHS0374
    4 28H  THICKNESS----------------- F10.4)                                  ASHS0375
 2009 FORMAT ( 25H1  MATERIAL PROPERTY DATA )                                 ASHS0376
 2010 FORMAT ( 1H1 5X 55HLEAST SQUARE EVALUATION OF ABOVE MATERIAL PROPEASHS0377
    1RTY DATA )                                                               ASHS0378
 2011 FORMAT ( 49H0 ERROR IN ABOVE INPUT DATA.  CHECK VALUE ON LINE I3 )ASHS0379
      END                                                                     ASHS0380
```

```
      SUBROUTINE TRIST(IX,JX,KX,TT,NNN)                                 ASHS0381
C                                                                       ASHS0382
      DIMENSION XX(5),YY(5),S(10,10),P(10),A(6,6),B(6,6),C(6,6),LM(3)   ASHS0383
     1 , HED(12),X(40,10),Y(40,10),T(40,10),CC(5,3),EE(6),U(40,10)      ASHS0384
     3 ,V(40,10)                                                        ASHS0385
C                                                                       ASHS0386
      COMMON KINN,KOUT,NMAX,MMAX,MB,NMT,NUMLD,NUMBC,TE,TR,              ASHS0387
     1 ES,XS,AS,TS,ESH,XSH,ASH,TSH,                                     ASHS0388
     1 HED,X,Y,T,U,V,CC,XX,YY,S,P,C                                     ASHS0389
C                                                                       ASHS0390
C     INITIALIZATION                                                    ASHS0391
C                                                                       ASHS0392
      AJ=XX(JX)-XX(IX)                                                  ASHS0393
      AK=XX(KX)-XX(IX)                                                  ASHS0394
      BJ=YY(JX)-YY(IX)                                                  ASHS0395
      BK=YY(KX)-YY(IX)                                                  ASHS0396
      TEMP=AJ*BK-BJ*AK                                                  ASHS0397
      AREA=TEMP/2.                                                      ASHS0398
      THICK=(XX(IX)+XX(JX)+XX(KX))/3.                                   ASHS0399
      DO 50 I=1,36                                                      ASHS0400
      A(I)=0.0                                                          ASHS0401
   50 B(I)=0.0                                                          ASHS0402
C                                                                       ASHS0403
C     CALCULATE MATERIAL PROPERTIES                                     ASHS0404
C                                                                       ASHS0405
      DO 60 I=1,3                                                       ASHS0406
   60 EE(I)=CC(1,I)+TT*(CC(2,I)+TT*(CC(3,I)+TT*(CC(4,I)+TT*CC(5,I))))   ASHS0407
      IF (NNN) 70,70,80                                                 ASHS0408
   70 EP=EE(1)                                                          ASHS0409
      ALP=EE(3)                                                         ASHS0410
      XNU=0.3155+0.000174*TT                                            ASHS0411
      GO TO 90                                                          ASHS0412
   80 EP=ES                                                             ASHS0413
      ALP=AS                                                            ASHS0414
      XNU=XS                                                            ASHS0415
   90 AT=ALP*(TT-TR)                                                    ASHS0416
      COMM=EP/(1.-2.*XNU)                                               ASHS0417
      TE=COMM*AT                                                        ASHS0418
      COMM=COMM/(1.+XNU)                                                ASHS0419
C                                                                       ASHS0420
C     FORM MATRIX (B)                                                   ASHS0421
C                                                                       ASHS0422
```

```
      E=XNU*COMM                                                        ASHS0423
      D=(1.-XNU)*COMM                                                   ASHS0424
      C=(.5-XNU)*COMM                                                   ASHS0425
      B(1,1)=D                                                          ASHS0426
      B(1,2)=E                                                          ASHS0427
      B(1,4)=E                                                          ASHS0428
      B(2,1)=E                                                          ASHS0429
      B(2,2)=D                                                          ASHS0430
      B(2,4)=E                                                          ASHS0431
      B(3,3)=C                                                          ASHS0432
      B(4,1)=E                                                          ASHS0433
      B(4,2)=E                                                          ASHS0434
      B(4,4)=D                                                          ASHS0435
C                                                                       ASHS0436
C     FORM MATRIX (A)                                                   ASHS0437
C                                                                       ASHS0438
      A(1,1)=BJ-BK                                                      ASHS0439
      A(1,3)=BK                                                         ASHS0440
      A(1,5)=-BJ                                                        ASHS0441
      A(2,2)=AK-AJ                                                      ASHS0442
      A(2,4)=-AK                                                        ASHS0443
      A(2,6)=AJ                                                         ASHS0444
      A(3,1)=AK-AJ                                                      ASHS0445
      A(3,2)=BJ-BK                                                      ASHS0446
      A(3,3)=-AK                                                        ASHS0447
      A(3,4)=BK                                                         ASHS0448
      A(3,5)=AJ                                                         ASHS0449
      A(3,6)=-BJ                                                        ASHS0450
      C=.333333*TEMP/THICK                                              ASHS0451
      A(4,1)=C                                                          ASHS0452
      A(4,3)=C                                                          ASHS0453
      A(4,5)=C                                                          ASHS0454
      DO 95 I=1,36                                                      ASHS0455
   95 A(I)=A(I)/TEMP                                                    ASHS0456
C                                                                       ASHS0457
C     FORM TRIANGULAR ELEMENT STIFFNESS MATRIX (A)T(B)(A)               ASHS0458
C                                                                       ASHS0459
      COMM=THICK*AREA                                                   ASHS0460
      DO 100 L=1,4                                                      ASHS0461
      DO 100 M=1,6                                                      ASHS0462
      C(L,M)=0.0                                                        ASHS0463
      DO 100 N=1,4                                                      ASHS0464
```

```
  100 C(L,M)=C(L,M)+B(L,N)*A(N,M)                               ASHS0465
      DO 210 L=1,6                                              ASHS0466
      DO 210 M=1,6                                              ASHS0467
      B(L,M)=0.0                                                ASHS0468
      DO 200 N=1,4                                              ASHS0469
  200 B(L,M)=B(L,M)+A(N,L)*C(N,M)                               ASHS0470
  210 B(L,M)=B(L,M)*COMM                                        ASHS0471
C                                                               ASHS0472
C     COMPUTE TEMPERATURE FORCES                                ASHS0473
C                                                               ASHS0474
      A(1)=AT*XX(IX)                                            ASHS0475
      A(2)=0.0                                                  ASHS0476
      A(3)=AT*XX(JX)                                            ASHS0477
      A(4)=AT*BJ                                                ASHS0478
      A(5)=AT*XX(KX)                                            ASHS0479
      A(6)=AT*BK                                                ASHS0480
C                                                               ASHS0481
      DO 220 I=1,6                                              ASHS0482
      EE(I)=0.0                                                 ASHS0483
      DO 220 K=1,6                                              ASHS0484
  220 EE(I)=EE(I)+B(I,K)*A(K)                                   ASHS0485
C                                                               ASHS0486
C     ADD ELEMENT STIFFNESS TO QUADRILATERAL STIFFNESS          ASHS0487
C                                                               ASHS0488
      LM(1)=2*IX-2                                              ASHS0489
      LM(2)=2*JX-2                                              ASHS0490
      LM(3)=2*KX-2                                              ASHS0491
      DO 250 I=1,3                                              ASHS0492
      DO 250 K=1,2                                              ASHS0493
      II=LM(I)+K                                                ASHS0494
      KK=2*I-2+K                                                ASHS0495
      P(II)=P(II)+EE(KK)                                        ASHS0496
      DO 250 J=1,3                                              ASHS0497
      DO 250 L=1,2                                              ASHS0498
      JJ=LM(J)+L                                                ASHS0499
      LL=2*J-2+L                                                ASHS0500
  250 S(II,JJ)=S(II,JJ)+B(KK,LL)                                ASHS0501
C                                                               ASHS0502
      RETURN                                                    ASHS0503
C                                                               ASHS0504
      END                                                       ASHS0505
```

```
      SUBROUTINE STRESS                                                 ASHS0506
C                                                                       ASHS0507
C                                                                       ASHS0508
      DIMENSION T(40,10),EE(3),CC(5,3),HED(12),                         ASHS0509
     1 XX(5),YY(5),S(10,10),P(10),X(40,10),Y(40,10),U(40,10),V(40,10)   ASHS0510
     2 ,A(9,9),B(9,2),C(6,6),D(9,9),E(9,2),SIGTT(40,10),SIGXM(40,10)    ASHS0511
     3 ,SIGXX(40,10),SIGYY(40,10),SIGXY(40,10),SIGNN(40,10),SIGSS(40,10)ASHS0512
     4 ,SIGNS(40,10),BSIGS(40),BSIGT(40),SIG(4)                         ASHS0513
C                                                                       ASHS0514
      COMMON KINN,KOUT,NMAX,MMAX,MB,NMT,NUMLD,NUMBC,TE,TR,              ASHS0515
     1 ES,XS,AS,TS,ESH,XSH,ASH,TSH,                                     ASHS0516
     1 HED,X,Y,T,U,V,CC,XX,YY,S,P,C                                     ASHS0517
     1 ,A,B,D,E,NN,MM,SIGXX,SIGYY,SIGXY,SIGMX,SIGMN,SIGTT,BSIGS,BSIGT   ASHS0518
     1  ,EE,SIGNN,SIGNS,SIGSS,SIGXM,SIG                                 ASHS0519
C                                                                       ASHS0520
C                                                                       ASHS0521
      NN=NMAX-1                                                         ASHS0522
      MM=MMAX-1                                                         ASHS0523
      DO 200 N=1,NN                                                     ASHS0524
      DO 200 M=1,MM                                                     ASHS0525
C                                                                       ASHS0526
      XX(1)=X(N,M)                                                      ASHS0527
      XX(2)=X(N,M+1)                                                    ASHS0528
      XX(3)=X(N+1,M)                                                    ASHS0529
      XX(4)=X(N+1,M+1)                                                  ASHS0530
      YY(1)=Y(N,M)                                                      ASHS0531
      YY(2)=Y(N,M+1)                                                    ASHS0532
      YY(3)=Y(N+1,M)                                                    ASHS0533
      YY(4)=Y(N+1,M+1)                                                  ASHS0534
      NNN=MB-M                                                          ASHS0535
      TT=.25*(T(N,M)+T(N,M+1)+T(N+1,M)+T(N+1,M+1))                      ASHS0536
C                                                                       ASHS0537
      CALL TRIST(1,4,2,TT,NNN)                                          ASHS0538
C                                                                       ASHS0539
      P(1)=U(N,M)                                                       ASHS0540
      P(2)=V(N,M)                                                       ASHS0541
      P(3)=U(N+1,M+1)                                                   ASHS0542
      P(4)=V(N+1,M+1)                                                   ASHS0543
      P(5)=U(N,M+1)                                                     ASHS0544
      P(6)=V(N,M+1)                                                     ASHS0545
C                                                                       ASHS0546
      DO 150 I=1,4                                                      ASHS0547
```

```
      SIG(I)=0.0                                                        ASHS0548
      DO 150 K=1,6                                                      ASHS0549
  150 SIG(I)=SIG(I)+C(I,K)*P(K)                                         ASHS0550
C                                                                       ASHS0551
      CALL TRIST(1,3,4,TT,NNN)                                          ASHS0552
C                                                                       ASHS0553
      P(1)=U(N,M)                                                       ASHS0554
      P(2)=V(N,M)                                                       ASHS0555
      P(3)=U(N+1,M)                                                     ASHS0556
      P(4)=V(N+1,M)                                                     ASHS0557
      P(5)=U(N+1,M+1)                                                   ASHS0558
      P(6)=V(N+1,M+1)                                                   ASHS0559
C                                                                       ASHS0560
      DO 170 I=1,4                                                      ASHS0561
      DO 160 K=1,6                                                      ASHS0562
  160 SIG(I)=SIG(I)+C(I,K)*P(K)                                         ASHS0563
  170 SIG(I)=SIG(I)/2.0                                                 ASHS0564
C                                                                       ASHS0565
      SIGXX(N,M)=SIG(1)-TE                                              ASHS0566
      SIGYY(N,M)=SIG(2)-TE                                              ASHS0567
      SIGXY(N,M)=SIG(3)                                                 ASHS0568
      SIGTT(N,M)=SIG(4)-TE                                              ASHS0569
C                                                                       ASHS0570
      COST = .5*( X(N,MB) + X(N+1,MB) - X(N,1) - X(N+1,1) ) / TS        ASHS0571
      SINT = .5*( Y(N,MB) + Y(N+1,MB) - Y(N,1) - Y(N+1,1) ) / TS        ASHS0572
      COS2=COST**2                                                      ASHS0573
      SIN2=SINT**2                                                      ASHS0574
      TT=2.0*SIGXY(N,M)*SINT*COST                                       ASHS0575
      SIGNN(N,M)=SIGXX(N,M)*COS2+SIGYY(N,M)*SIN2+TT                     ASHS0576
      SIGSS(N,M)=SIGXX(N,M)*SIN2+SIGYY(N,M)*COS2-TT                     ASHS0577
      SIGNS(N,M)=(SIGYY(N,M)-SIGXX(N,M))*SINT*COST+SIGXY(N,M)*          ASHS0578
     1 (COS2-SIN2)                                                      ASHS0579
  200 CONTINUE                                                          ASHS0580
C                                                                       ASHS0581
C     PRINT STRESSES                                                    ASHS0582
C                                                                       ASHS0583
      WRITE OUTPUT TAPE KOUT,1006                                       ASHS0584
      CALL PRINTM(SIGTT,NN,MM,40)                                       ASHS0585
      WRITE OUTPUT TAPE KOUT,1007                                       ASHS0586
      CALL PRINTM(SIGXX,NN,MM,40)                                       ASHS0587
      WRITE OUTPUT TAPE KOUT,1008                                       ASHS0588
      CALL PRINTM(SIGYY,NN,MM,40)                                       ASHS0589
```

```
      WRITE OUTPUT TAPE KOUT,1009                                       ASHS0590
      CALL PRINTM(SIGXY,NN,MM,40)                                       ASHS0591
      WRITE OUTPUT TAPE KOUT,1011                                       ASHS0592
      CALL PRINTM(SIGSS,NN,MM,40)                                       ASHS0593
C                                                                       ASHS0594
C     CALCULATION OF STRESSES IN SANDWICH PLATES                        ASHS0595
C                                                                       ASHS0596
      ED=ESH/(1.-XSH**2)                                                ASHS0597
      EE=XSH*ED                                                         ASHS0598
      MM=MB-1                                                           ASHS0599
C                                                                       ASHS0600
      DO 225 N=1,NN                                                     ASHS0601
      DO 225 M=1,MB,MM                                                  ASHS0602
      RI=X(N,M)                                                         ASHS0603
      ZI=Y(N,M)                                                         ASHS0604
      RJ=X(N+1,M)                                                       ASHS0605
      ZJ=Y(N+1,M)                                                       ASHS0606
      AA=ZJ-ZI                                                          ASHS0607
      BB=RJ-RI                                                          ASHS0608
      XL2=AA**2+BB**2                                                   ASHS0609
      AT =.5*ASH*(T(N,M)+T(N+1,M))                                      ASHS0610
      EPT=(U(N,M)+U(N+1,M))/(RI+RJ)-AT                                  ASHS0611
      EPS=(BB*(U(N+1,M)-U(N,M))+AA*(V(N+1,M)-V(N,M)))/XL2 -AT           ASHS0612
      SIGTT(N,M)=ED*EPT+EE*EPS                                          ASHS0613
  225 SIGSS(N,M)=EE*EPT+ED*EPS                                          ASHS0614
C                                                                       ASHS0615
      WRITE OUTPUT TAPE KOUT, 1013,                                     ASHS0616
     1( N, SIGTT(N,1), SIGSS(N,1), SIGTT(N,MB), SIGSS(N,MB), N=1,NN )   ASHS0617
C                                                                       ASHS0618
C     CALCULATION OF STRESSES IN BOND LAYER                             ASHS0619
C                                                                       ASHS0620
      DO 300 N=1,NN                                                     ASHS0621
      TT = 0.50*( T(N,MB)+T(N+1,MB) )                                   ASHS0622
      DO 260 I=1,3                                                      ASHS0623
  260 EE(I)=CC(1,I)+TT*(CC(2,I)+TT*(CC(3,I)+TT*(CC(4,I)+TT*CC(5,I))))   ASHS0624
      XNU = 0.3155+0.000174*TT                                          ASHS0625
      AA = Y(N+1,MB)-Y(N,MB)                                            ASHS0626
      BB = X(N+1,MB)-X(N,MB)                                            ASHS0627
      XL2 = AA**2+BB**2                                                 ASHS0628
      EEE = XNU*SIGNN(N,MB)/EE(2) - EE(3)*TT                            ASHS0629
      EPT = ( U(N,MB)+U(N+1,MB) )/( X(N+1,MB)+X(N,MB) ) + EEE           ASHS0630
      EPS = (BB*(U(N+1,MB)-U(N,MB))+AA*(V(N+1,MB)-V(N,MB)))/XL2+EEE     ASHS0631
```

```
      COMM = EE(2)/(1.-XNU**2)                                          ASHS0632
      BSIGS(N) = COMM*(EPS+XNU*EPT)                                     ASHS0633
  300 BSIGT(N) = COMM*(EPT+XNU*EPS)                                     ASHS0634
      WRITE OUTPUT TAPE KOUT, 1014, (N, BSIGS(N), BSIGT(N), SIGNN(N,MB),ASHS0635
     1 SIGNS(N,MB), N=1,NN )                                            ASHS0636
C                                                                       ASHS0637
  250 RETURN                                                            ASHS0638
C                                                                       ASHS0639
C     FORMAT STATEMENTS                                                 ASHS0640
C                                                                       ASHS0641
 1006 FORMAT (10H1 T-STRESS)                                            ASHS0642
 1007 FORMAT (10H1 R-STRESS)                                            ASHS0643
 1008 FORMAT (10H1 Z-STRESS)                                            ASHS0644
 1009 FORMAT (10H1RZ-STRESS)                                            ASHS0645
 1011 FORMAT (10H1 S-STRESS)                                            ASHS0646
 1013 FORMAT ( 1H1 42X 27HSTRESSES IN SANDWICH PLATES /                 ASHS0647
     1          27X 11HLOWER PLATE 39X 11HUPPER PLATE /                 ASHS0648
     2          20X 5HMERID 15X 4HHOOP 25X 5HMERID 15X 4HHOOP /         ASHS0649
     3  4X 1HN 14X 6HSTRESS 14X 6HSTRESS 24X 6HSTRESS 14X 6HSTRESS //   ASHS0650
     4    ( I5, 2F20.0, 10X 2F20.0) )                                   ASHS0651
 1014 FORMAT (1H1 11X 22HSTRESSES IN BOND LAYER //                      ASHS0652
     145H    N     MERID      HOOP     NORMAL     SHEAR //              ASHS0653
     2(I5, 4F10.1) )                                                    ASHS0654
      END                                                               ASHS0655
```

```
      SUBROUTINE LEAST(A,B,C,D,E,N,M,L)                                 ASHS0656
C                                                                       ASHS0657
      DIMENSION A(50,5),B(50,3),C(5,5),D(5,3),E(5,3)                    ASHS0658
C                                                                       ASHS0659
C     (C)=(A)T*(A)  AND (D)=(A)T*(B)                                    ASHS0660
C                                                                       ASHS0661
      DO 200 I=1,M                                                      ASHS0662
      DO 100 J=1,M                                                      ASHS0663
      C(I,J)=0.0                                                        ASHS0664
      DO 100 K=1,N                                                      ASHS0665
  100 C(I,J)=C(I,J)+A(K,I)*A(K,J)                                       ASHS0666
      DO 200 J=1,L                                                      ASHS0667
      D(I,J)=0.0                                                        ASHS0668
      DO 200 K=1,N                                                      ASHS0669
  200 D(I,J)=D(I,J)+A(K,I)*B(K,J)                                       ASHS0670
C                                                                       ASHS0671
C     INVERT (C)                                                        ASHS0672
C                                                                       ASHS0673
      CALL INVERT(C,M,5,B,E)                                            ASHS0674
C                                                                       ASHS0675
C      (E)=(C)*(D)                                                      ASHS0676
C                                                                       ASHS0677
      DO 300 I=1,M                                                      ASHS0678
      DO 300 J=1,L                                                      ASHS0679
      E(I,J)=0.0                                                        ASHS0680
      DO 300 K=1,M                                                      ASHS0681
  300 E(I,J)=E(I,J)+C(I,K)*D(K,J)                                       ASHS0682
C                                                                       ASHS0683
C     CHECK RESULTS  (B)=(A)*(E)                                        ASHS0684
C                                                                       ASHS0685
      DO 400 I=1,N                                                      ASHS0686
      DO 400 J=1,L                                                      ASHS0687
      B(I,J)=0.0                                                        ASHS0688
      DO 400 K=1,M                                                      ASHS0689
  400 B(I,J)=B(I,J)+A(I,K)*E(K,J)                                       ASHS0690
C                                                                       ASHS0691
      RETURN                                                            ASHS0692
C                                                                       ASHS0693
      END                                                               ASHS0694
```

```
      SUBROUTINE SYMSOL (A,B,NN,MM)
C
      DIMENSION A(800,24),B(800),C(24)
C
      N = 0
  100 N = N+1
C
C     REDUCE N TH EQUATION
C
C        1. DIVIDE RIGHT SIDE BY DIAGONAL ELEMENT
C
      B(N) = B(N) / A(N,1)
C
C        2. CHECK FOR LAST EQUATION
C
      IF(N-NN) 150,300,150
C
C        3. DIVIDE N TH EQUATION BY DIAGONAL ELEMENT
C
  150 DO 200 K=2,MM
      C(K) = A(N,K)
  200 A(N,K) = A(N,K) / A(N,1)
C
C        4. REDUCE REMAINING EQUATIONS
C
      DO 260 L=2,MM
      I = N+L-1
      IF(NN-I) 260,240,240
  240 J=0
      DO 250 K=L,MM
      J=J+1
  250 A(I,J) = A(I,J) - C(L) * A(N,K)
      B(I) = B(I) - C(L) * B(N)
  260 CONTINUE
      GO TO 100
C
C     BACK SUBSTITUTION
C
  300 N = N-1
C
C        1. CHECK FOR FIRST EQUATION
C
```

```
      IF(N) 350,500,350                                                 ASHS0825
C                                                                       ASHS0826
C        2. CALCULATE UNKNOWN B(N)                                      ASHS0827
C                                                                       ASHS0828
  350 DO 400 K=2,MM                                                     ASHS0829
      L = N+K-1                                                         ASHS0830
      IF(NN-L) 400,370,370                                              ASHS0831
  370 B(N) = B(N) - A(N,K) * B(L)                                       ASHS0832
  400 CONTINUE                                                          ASHS0833
      GO TO 300                                                         ASHS0834
C                                                                       ASHS0835
  500 RETURN                                                            ASHS0836
C                                                                       ASHS0837
      END                                                               ASHS0838
```

```
      SUBROUTINE INVERT(A,NN,N,M,C)                                     ASHS0695
C                                                                       ASHS0696
C            GENERAL MATRIX INVERSION SUBROUTINE                        ASHS0697
C                                                                       ASHS0698
      DIMENSION A(1),M(1),C(1)                                          ASHS0699
C                                                                       ASHS0700
      DO 90 I=1,NN                                                      ASHS0701
   90 M(I)=-I                                                           ASHS0702
C                                                                       ASHS0703
      DO 140 I=1,NN                                                     ASHS0704
C                                                                       ASHS0705
C     LOCATE LARGEST ELEMENT                                            ASHS0706
C                                                                       ASHS0707
      D=0.0                                                             ASHS0708
      DO 112 L=1,NN                                                     ASHS0709
      IF (M(L)) 100,100,112                                             ASHS0710
  100 J=L                                                               ASHS0711
      DO 110 K=1,NN                                                     ASHS0712
      IF (M(K)) 103,103,108                                             ASHS0713
  103 IF (ABSF(D)-ABSF(A(J))) 105,105,108                               ASHS0714
  105 LD=L                                                              ASHS0715
      KD=K                                                              ASHS0716
      D=A(J)                                                            ASHS0717
  108 J=J+N                                                             ASHS0718
  110 CONTINUE                                                          ASHS0719
  112 CONTINUE                                                          ASHS0720
C                                                                       ASHS0721
C     INTERCHANGE ROWS                                                  ASHS0722
C                                                                       ASHS0723
      TEMP=-M(LD)                                                       ASHS0724
      M(LD)=M(KD)                                                       ASHS0725
      M(KD)=TEMP                                                        ASHS0726
      L=LD                                                              ASHS0727
      K=KD                                                              ASHS0728
      DO 114 J=1,NN                                                     ASHS0729
      C(J)=A(L)                                                         ASHS0730
      A(L)=A(K)                                                         ASHS0731
      A(K)=C(J)                                                         ASHS0732
      L=L+N                                                             ASHS0733
  114 K=K+N                                                             ASHS0734
C                                                                       ASHS0735
C     DIVIDE COLUMN BY LARGEST ELEMENT                                  ASHS0736
```

```
C                                                                       ASHS0737
      NR=(KD-1)*N+1                                                     ASHS0738
      NH=NR+N-1                                                         ASHS0739
      DO 115 K=NR,NH                                                    ASHS0740
  115 A(K)=A(K)/D                                                       ASHS0741
C                                                                       ASHS0742
C     REDUCE REMAINING ROWS AND COLUMNS                                 ASHS0743
C                                                                       ASHS0744
      L=1                                                               ASHS0745
      DO 135 J=1,NN                                                     ASHS0746
      IF (J-KD) 130,125,130                                             ASHS0747
  125 L=L+N                                                             ASHS0748
      GO TO 135                                                         ASHS0749
  130 DO 134 K=NR,NH                                                    ASHS0750
      A(L)=A(L)-C(J)*A(K)                                               ASHS0751
  134 L=L+1                                                             ASHS0752
  135 CONTINUE                                                          ASHS0753
C                                                                       ASHS0754
C     REDUCE ROW                                                        ASHS0755
C                                                                       ASHS0756
      C(KD)=-1.0                                                        ASHS0757
      J=KD                                                              ASHS0758
      DO 140 K=1,NN                                                     ASHS0759
      A(J)=-C(K)/D                                                      ASHS0760
  140 J=J+N                                                             ASHS0761
C                                                                       ASHS0762
C     INTERCHANGE COLUMNS                                               ASHS0763
C                                                                       ASHS0764
      DO 200 I=1,NN                                                     ASHS0765
      L=0                                                               ASHS0766
  150 L=L+1                                                             ASHS0767
      IF(M(L)-I) 150,160,150                                            ASHS0768
  160 K=(L-1)*N+1                                                       ASHS0769
      J=(I-1)*N+1                                                       ASHS0770
      M(L)=M(I)                                                         ASHS0771
      M(I)=I                                                            ASHS0772
      DO 200 L=1,NN                                                     ASHS0773
      TEMP=A(K)                                                         ASHS0774
      A(K)=A(J)                                                         ASHS0775
      A(J)=TEMP                                                         ASHS0776
      J=J+1                                                             ASHS0777
  200 K=K+1                                                             ASHS0778
```

```
C                                                                 ASHS0779
      RETURN                                                      ASHS0780
C                                                                 ASHS0781
      END                                                         ASHS0782
```

```
      SUBROUTINE PRINTM (A,NR,NC,MAXR)                                  ASHS0839
C     SUBROUTINE TO PRINT ANY ARRAY                                     ASHS0840
      DIMENSION A(1),NHED(10)                                           ASHS0841
      COMMON KINN,KOUT                                                  ASHS0842
C                                                                       ASHS0843
      DO 50 I=1,NC,10                                                   ASHS0844
      II=NC-I+1                                                         ASHS0845
      IF (II-10) 20,20,10                                               ASHS0846
   10 II=10                                                             ASHS0847
   20 DO 30 J=1,II                                                      ASHS0848
   30 NHED(J)=I+J-1                                                     ASHS0849
C                                                                       ASHS0850
      WRITE OUTPUT TAPE KOUT,120,(NHED(J),J=1,II)                       ASHS0851
C                                                                       ASHS0852
      DO 50 J=1,NR                                                      ASHS0853
      KL=J+(I-1)*MAXR                                                   ASHS0854
      KH=KL+(II-1)*MAXR                                                 ASHS0855
   50 WRITE OUTPUT TAPE KOUT,130,(J,(A(K),K=KL,KH,MAXR))                ASHS0856
C                                                                       ASHS0857
      RETURN                                                            ASHS0858
C                                                                       ASHS0859
  120 FORMAT (8H0   N/M  10I11)                                         ASHS0860
  130 FORMAT (I5,3X,10F11.3)                                            ASHS0861
      END                                                               ASHS0862
```

APPENDIX F

PROGRAM LISTING - NON-AXISYMMETRIC HEAT SHIELDS

```
CNAHS     NON-AXISYMMETRIC ANALYSIS HEAT SHIELDS                          NAHS0001
C                                                                         NAHS0002
      DIMENSION R(30,8),Z(30,8),T(30,8),TC(30,8),E(10,10),CC(10,10),     NAHS0003
     1 XX(5),YY(5),S(15,15),P(15),KODE(30,10),                            NAHS0004
     2 SS(720,30),RR(720),NTAG(3),C(9,9),B(9,9),G(9,9)                    NAHS0005
      COMMON KINN,KOUT,N,M,L,NMAX,MMAX,MB,NMT,NH,NB,TE,TTT,ES,XS,AS,TS,  NAHS0006
     1 ESH,XSH,ASH,TSH,R,Z,T,TC,E,CC,XX,YY,S,P,C,B,G,KODE,RR,SS          NAHS0007
C                                                                         NAHS0008
C                                                                         NAHS0009
   50 KINN=5                                                              NAHS0010
      KOUT=6                                                              NAHS0011
      REWIND 20                                                           NAHS0012
C                                                                         NAHS0013
      CALL INPUT                                                          NAHS0014
C                                                                         NAHS0015
      MX=3*MMAX                                                           NAHS0016
      NS=NMAX*MMAX                                                        NAHS0017
      MM=MMAX-1                                                           NAHS0018
      NN=NMAX-1                                                           NAHS0019
      MBAND=3*(MMAX+2)                                                    NAHS0020
      NEQ=3*NS                                                            NAHS0021
      MBB=MB-1                                                            NAHS0022
      MCC=MB+1                                                            NAHS0023
C                                                                         NAHS0024
C     READ IN ADDITIONAL BOUNDARY CONDITIONS                              NAHS0025
C                                                                         NAHS0026
      DO 60 N=1,NMAX                                                      NAHS0027
      DO 60 M=1,MMAX                                                      NAHS0028
   60 KODE(N,M)=0                                                         NAHS0029
      WRITE OUTPUT TAPE KOUT,2000                                         NAHS0030
      DO 70 K=1,NB                                                        NAHS0031
      READ INPUT TAPE KINN, 1000, N,M,KK                                  NAHS0032
      KODE(N,M)=KK                                                        NAHS0033
      WRITE OUTPUT TAPE KOUT,2001, N,M,KODE(N,M)                          NAHS0034
   70 CONTINUE                                                            NAHS0035
C                                                                         NAHS0036
C     SOLVE STRUCTURE FOR EACH HARMONIC                                   NAHS0037
C                                                                         NAHS0038
      DO 500 L=1,NH                                                       NAHS0039
      WRITE OUTPUT TAPE KOUT,2006, L                                      NAHS0040
C                                                                         NAHS0041
C     COMPUTE FOURIER COEFFICIENTS FOR HARMONIC L                         NAHS0042
C                                                                         NAHS0043
      XL=0.0                                                              NAHS0044
```

```
      TT=0.0                                                            NAHS0045
      IF(L-1) 102,101,102                                               NAHS0046
  101 TT=TE                                                             NAHS0047
  102 DO 105 N=1,NMAX                                                   NAHS0048
      TC(N,MMAX)=TT                                                     NAHS0049
      TC(N,MB)=E(1,L)+XL*(E(2,L)+XL*(E(3,L)+XL*(E(4,L)+XL*(E(5,L)       NAHS0050
     1 +XL*E(6,L)))))                                                   NAHS0051
      XL=XL+SQRTF((R(N+1,MB)-R(N,MB))**2+(Z(N+1,MB)-Z(N,MB))**2)        NAHS0052
      DO 104 M=1,MBB                                                    NAHS0053
  104 TC(N,M)=TC(N,MB)                                                  NAHS0054
      TXX=(TC(N,MMAX)-TC(N,MB))/FLOATF(MMAX-MB)**2                      NAHS0055
      DO 105 M=MCC,MMAX                                                 NAHS0056
  105 TC(N,M)=TC(N,MB)+TXX*FLOATF(M-MB)**2                              NAHS0057
C                                                                       NAHS0058
      IF(L-1) 106,106,108                                               NAHS0059
  106 DO 107 N=1,NMAX                                                   NAHS0060
      DO 107 M=1,MMAX                                                   NAHS0061
  107 TC(N,M)=TC(N,M)-TTT                                               NAHS0062
  108 CALL PRINTM(TC,NMAX,MMAX,30)                                      NAHS0063
C                                                                       NAHS0064
C     FORMATION OF STIFFNESS ARRAY                                      NAHS0065
C                                                                       NAHS0066
      DO 175 I=1,NEQ                                                    NAHS0067
      DO 170 J=1,MBAND                                                  NAHS0068
  170 SS(I,J)=0.0                                                       NAHS0069
  175 RR(I)=0.0                                                         NAHS0070
C                                                                       NAHS0071
      DO 200 N=1,NN                                                     NAHS0072
      DO 200 M=1,MM                                                     NAHS0073
C                                                                       NAHS0074
      DO 180 I=1,15                                                     NAHS0075
      P(I)=0.0                                                          NAHS0076
      DO 180 J=1,15                                                     NAHS0077
  180 S(I,J)=0.0                                                        NAHS0078
C                                                                       NAHS0079
      XX(1)=R(N,M)                                                      NAHS0080
      XX(2)=R(N,M+1)                                                    NAHS0081
      XX(3)=R(N+1,M)                                                    NAHS0082
      XX(4)=R(N+1,M+1)                                                  NAHS0083
      YY(1)=Z(N,M)                                                      NAHS0084
      YY(2)=Z(N,M+1)                                                    NAHS0085
      YY(3)=Z(N+1,M)                                                    NAHS0086
      YY(4)=Z(N+1,M+1)                                                  NAHS0087
C                                                                       NAHS0088
```

```
      CALL STIFFT(1,4,2)                                                NAHS0089
      CALL STIFFT(1,3,4)                                                NAHS0090
C                                                                       NAHS0091
C       ADD QUADRILATERAL STIFFNESS TO TOTAL STIFFNESS                  NAHS0092
C                                                                       NAHS0093
      DO 200 I=1,6                                                      NAHS0094
      II=((N-1)*MMAX+M-1)*3+I                                           NAHS0095
      KK=II+MX                                                          NAHS0096
      RR(II)=RR(II)+P(I)                                                NAHS0097
      RR(KK)=RR(KK)+P(I+6)                                              NAHS0098
      DO 190 J=I,6                                                      NAHS0099
      JJ=J-I+1                                                          NAHS0100
      SS(II,JJ)=SS(II,JJ)+S(I,J)                                        NAHS0101
  190 SS(KK,JJ)=SS(KK,JJ)+S(I+6,J+6)                                    NAHS0102
      DO 200 J=1,6                                                      NAHS0103
      JJ=J-I+1+MX                                                       NAHS0104
  200 SS(II,JJ)=SS(II,JJ)+S(I,J+6)                                      NAHS0105
C                                                                       NAHS0106
C     STIFFNESS OF PLATE ELEMENTS                                       NAHS0107
C                                                                       NAHS0108
      MBB=MB-1                                                          NAHS0109
      DO 600 N=1,NN                                                     NAHS0110
      DO 600 M=1,MB,MBB                                                 NAHS0111
C                                                                       NAHS0112
      CALL STIFFP                                                       NAHS0113
C                                                                       NAHS0114
      DO 600 I=1,3                                                      NAHS0115
      II=((N-1)*MMAX+M-1)*3+I                                           NAHS0116
      KK=II+MX                                                          NAHS0117
      RR(II)=RR(II)+P(I)                                                NAHS0118
      RR(KK)=RR(KK)+P(I+3)                                              NAHS0119
      DO 590 J=I,3                                                      NAHS0120
      JJ=J-I+1                                                          NAHS0121
      SS(II,JJ)=SS(II,JJ)+S(I,J)                                        NAHS0122
  590 SS(KK,JJ)=SS(KK,JJ)+S(I+3,J+3)                                    NAHS0123
      DO 600 J=1,3                                                      NAHS0124
      JJ=J-I+1+MX                                                       NAHS0125
  600 SS(II,JJ)=SS(II,JJ)+S(I,J+3)                                      NAHS0126
C                                                                       NAHS0127
C     SET BOUNDARY CONDITIONS ALONG AXIS FOR HARMONIC L                 NAHS0128
C                                                                       NAHS0129
      DO 208 M=1,MMAX                                                   NAHS0130
      IF(L-2) 205,206,207                                               NAHS0131
  205 KODE(1,M)=4                                                       NAHS0132
```

```
      GO TO 208                                                         NAHS0133
  206 KODE(1,M)=3                                                       NAHS0134
      GO TO 208                                                         NAHS0135
  207 KODE(1,M)=7                                                       NAHS0136
  208 CONTINUE                                                          NAHS0137
C                                                                       NAHS0138
C       MODIFY STIFFNESS MATRIX                                         NAHS0139
C                                                                       NAHS0140
      K=1                                                               NAHS0141
      DO 300 N=1,NMAX                                                   NAHS0142
      DO 300 M=1,MMAX                                                   NAHS0143
      NTAG(1)=0                                                         NAHS0144
      NTAG(2)=0                                                         NAHS0145
      NTAG(3)=0                                                         NAHS0146
      KK=KODE(N,M)                                                      NAHS0147
      IF(KK) 250,250,210                                                NAHS0148
  210 GO TO (230,220,245,215,225,240,235),KK                            NAHS0149
  215 NTAG(1)=K                                                         NAHS0150
  220 NTAG(2)=K+1                                                       NAHS0151
      GO TO 250                                                         NAHS0152
  225 NTAG(3)=K+2                                                       NAHS0153
  230 NTAG(1)=K                                                         NAHS0154
      GO TO 250                                                         NAHS0155
  235 NTAG(1)=K                                                         NAHS0156
  240 NTAG(2)=K+1                                                       NAHS0157
  245 NTAG(3)=K+2                                                       NAHS0158
  250 K=K+3                                                             NAHS0159
      DO 300 LL=1,3                                                     NAHS0160
      II=NTAG(LL)                                                       NAHS0161
      IF(II)300,300,260                                                 NAHS0162
  260 DO 270 J=1,MBAND                                                  NAHS0163
      SS(II,J)=0.0                                                      NAHS0164
      III=II+1-J                                                        NAHS0165
      IF(III) 270,270,265                                               NAHS0166
  265 SS(III,J)=0.0                                                     NAHS0167
  270 CONTINUE                                                          NAHS0168
      SS(II,1)=1.0                                                      NAHS0169
      RR(II)=0.0                                                        NAHS0170
  300 CONTINUE                                                          NAHS0171
C                                                                       NAHS0172
C     SOLVE FOR DISPLACEMENTS                                           NAHS0173
C                                                                       NAHS0174
C                                                                       NAHS0175
      CALL SYMSOL(SS,RR,NEQ,MBAND)                                      NAHS0176
```

```
C                                                                       NAHS0177
      CALL STRESS                                                       NAHS0178
C                                                                       NAHS0179
  500 CONTINUE                                                          NAHS0180
C                                                                       NAHS0181
      CALL OUTPUT                                                       NAHS0182
C                                                                       NAHS0183
C     FORMAT STATEMENTS                                                 NAHS0184
C                                                                       NAHS0185
 1000 FORMAT (3I5)                                                      NAHS0186
 2000 FORMAT (20H1BOUNDARY CONDITIONS // 18H     N     M  CODE )        NAHS0187
 2001 FORMAT (3I6)                                                      NAHS0188
 2006 FORMAT (3H1L=I3)                                                  NAHS0189
C                                                                       NAHS0190
      END                                                               NAHS0191
```

```
      SUBROUTINE INPUT                                                  NAHS0192
C                                                                       NAHS0193
C     COMMON AND DIMENSION STATEMENTS                                   NAHS0194
C                                                                       NAHS0195
      DIMENSION R(30,8),Z(30,8),T(30,8),TC(30,8),E(10,10),CC(10,10),    NAHS0196
     1 RR(10),ZZ(10),TA(30),TT(50),A(50,10),B(10,10),D(10,10),          NAHS0197
     2 EE(50,10),HED(12)                                                NAHS0198
C                                                                       NAHS0199
      COMMON KINN,KOUT,N,M,L,NMAX,MMAX,MB,NMT,NH,NB,TE,TTT,ES,XS,AS,TS, NAHS0200
     1 ESH,XSH,ASH,TSH,R,Z,T,TC,E,CC,RR,ZZ,TA,TT,A,B,D,EE,HED           NAHS0201
C                                                                       NAHS0202
C     READ AND PRINT INPUT DATA                                         NAHS0203
C                                                                       NAHS0204
      READ INPUT TAPE KINN,1000,                                        NAHS0205
     1 HED,NMAX,MMAX,MB,NMT,NH,NB,TE,TTT,ES,XS,AS,TS,ESH,XSH,ASH,TSH    NAHS0206
      WRITE OUTPUT TAPE KOUT,2000, HED,NMAX,MMAX,MB,NMT,NH,NB,TE,TTT    NAHS0207
      WRITE OUTPUT TAPE KOUT,2006                                       NAHS0208
      WRITE OUTPUT TAPE KOUT,2008, ES,XS,AS,TS                          NAHS0209
      WRITE OUTPUT TAPE KOUT,2007                                       NAHS0210
      WRITE OUTPUT TAPE KOUT,2008, ESH,XSH,ASH,TSH                      NAHS0211
      READ INPUT TAPE KINN,1001,                                        NAHS0212
     1 (R(N,MB),Z(N,MB),T(N,MB),TA(N),N=1,NMAX)                         NAHS0213
      WRITE OUTPUT TAPE KOUT,2001,                                      NAHS0214
     1 (R(N,MB),Z(N,MB),T(N,MB),TA(N),N=1,NMAX)                         NAHS0215
C                                                                       NAHS0216
C     LEAST SQUARE EVALUATION OF MATERIAL PROPERTIES                    NAHS0217
C                                                                       NAHS0218
      READ INPUT TAPE KINN,1002,                                        NAHS0219
     1 (TT(I),EE(I,1),EE(I,2),EE(I,3),I=1,NMT)                          NAHS0220
      WRITE OUTPUT TAPE KOUT, 2011                                      NAHS0221
      WRITE OUTPUT TAPE KOUT,2002,                                      NAHS0222
     1 (TT(I),EE(I,1),EE(I,2),EE(I,3),I=1,NMT)                          NAHS0223
C                                                                       NAHS0224
      DO 50 I=1,NMT                                                     NAHS0225
      A(I,1)=1.0                                                        NAHS0226
      A(I,2)=TT(I)                                                      NAHS0227
      A(I,3)=TT(I)*TT(I)                                                NAHS0228
      A(I,4)=A(I,3)*TT(I)                                               NAHS0229
   50 A(I,5)=A(I,4)*TT(I)                                               NAHS0230
C                                                                       NAHS0231
      CALL LEAST(A,EE,B,D,CC,NMT,5,3)                                   NAHS0232
C                                                                       NAHS0233
      WRITE OUTPUT TAPE KOUT, 2012                                      NAHS0234
      WRITE OUTPUT TAPE KOUT,2002,                                      NAHS0235
```

```
     1 (TT(I),EE(I,1),EE(I,2),EE(I,3),I=1,NMT)                          NAHS0236
C                                                                       NAHS0237
C     GENERATE MESH                                                     NAHS0238
C                                                                       NAHS0239
      MC = MB+1                                                         NAHS0240
      MBB = MB-1                                                        NAHS0241
      SHL = FLOATF(MBB)                                                 NAHS0242
      ABL = FLOATF(MMAX-MB)                                             NAHS0243
      DO 200 N=1,NMAX                                                   NAHS0244
C                                                                       NAHS0245
C     CHECK FOR END POINTS                                              NAHS0246
C                                                                       NAHS0247
      NL = N-1                                                          NAHS0248
      NNH=N+1                                                           NAHS0249
      IF(NL)160,150,160                                                 NAHS0250
  150 SS = 0.                                                           NAHS0251
      CX = 1.                                                           NAHS0252
      GO TO 185                                                         NAHS0253
  160 IF(N-NMAX)180,170,180                                             NAHS0254
  170 NNH=N                                                             NAHS0255
C                                                                       NAHS0256
C     COMPUTE SIN AND COS AT EACH POINT ON BOND LINE                    NAHS0257
C                                                                       NAHS0258
  180 XX = R( NNH,MB) - R(NL,MB)                                        NAHS0259
      YY = Z(NNH ,MB) - Z(NL,MB)                                        NAHS0260
      ZZ = SQRTF(XX**2 + YY**2)                                         NAHS0261
      SS = YY/ZZ                                                        NAHS0262
      CX = XX/ZZ                                                        NAHS0263
C                                                                       NAHS0264
C     COMPUTE COORDINATES OF POINTS IN CONSTANT THICKNESS SHELL         NAHS0265
C                                                                       NAHS0266
  185 DO 190 M=1,MBB                                                    NAHS0267
      TT = TS*FLOATF(MB-M)/SHL                                          NAHS0268
      T(N,M) = T(N,MB)                                                  NAHS0269
      R(N,M) = R(N,MB) + TT*SS                                          NAHS0270
  190 Z(N,M) = Z(N,MB) - TT*CX                                          NAHS0271
C                                                                       NAHS0272
C     COMPUTE COORDINATES OF POINTS IN VARIABLE THICKNESS ABLATER       NAHS0273
C                                                                       NAHS0274
      DO 200 M=MC,MMAX                                                  NAHS0275
      TT = TA(N)*FLOATF(M-MB)/ABL                                       NAHS0276
      T(N,M) = T(N,MB) + (TE - T(N,MB)) * (TT/TA(N))**2                 NAHS0277
      R(N,M) = R(N,MB) - TT*SS                                          NAHS0278
  200 Z(N,M) = Z(N,MB) + TT*CX                                          NAHS0279
```

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C                                                                       NAHS0280
C     DUMP OUT R-ORDINATE TABLE AND Z-ORDINATE TABLE                    NAHS0281
C                                                                       NAHS0282
      WRITE OUTPUT TAPE KOUT,2003                                       NAHS0283
      CALL PRINTM (R,NMAX,MMAX,30)                                      NAHS0284
      WRITE OUTPUT TAPE KOUT,2004                                       NAHS0285
      CALL PRINTM (Z,NMAX,MMAX,30)                                      NAHS0286
      WRITE OUTPUT TAPE KOUT,2005                                       NAHS0287
      CALL PRINTM (T,NMAX,MMAX,30)                                      NAHS0288
C                                                                       NAHS0289
C     READ IN THREE-DIMENSIONAL TEMPERATURE DISTRIBUTION                NAHS0290
C                                                                       NAHS0291
      TT(1)=0.0                                                         NAHS0292
      DO 240 I=2,19                                                     NAHS0293
  240 TT(I)=TT(I-1)+10.                                                 NAHS0294
      READ INPUT TAPE KINN, 1003,((RR(J),J=1,9),(ZZ(J),J=1,9))          NAHS0295
      WRITE OUTPUT TAPE KOUT,2009,((RR(J),J=1,9),(ZZ(J),J=1,9))         NAHS0296
      READ INPUT TAPE KINN,  1003,((EE(I,J),J=1,9),I=1,19)              NAHS0297
      WRITE OUTPUT TAPE KOUT, 2013                                      NAHS0298
      WRITE OUTPUT TAPE KOUT,2010,(TT(I),(EE(I,J),J=1,9),I=1,19)        NAHS0299
C                                                                       NAHS0300
      DO 250 I=1,19                                                     NAHS0301
      XX=0.17453*FLOATF(I-1)                                            NAHS0302
      DO 250 J=1,NH                                                     NAHS0303
      YY=XX*FLOATF(J-1)                                                 NAHS0304
  250 A(I,J)=COSF(YY)                                                   NAHS0305
C                                                                       NAHS0306
      DO 270 I=1,9                                                      NAHS0307
      DO 260 J=1,NH                                                     NAHS0308
      E(J,I)=.5*(EE(1,I)*A(1,J)+EE(19,I)*A(19,J))                       NAHS0309
      DO 255 K=2,18                                                     NAHS0310
  255 E(J,I)=E(J,I)+EE(K,I)*A(K,J)                                      NAHS0311
  260 E(J,I)=E(J,I)/9.                                                  NAHS0312
  270 E(1,I)=E(1,I)/2.                                                  NAHS0313
C                                                                       NAHS0314
      DO 280 I=1,19                                                     NAHS0315
      DO 280 J=1,9                                                      NAHS0316
      EE(I,J)=0.0                                                       NAHS0317
      DO 280 K=1,NH                                                     NAHS0318
  280 EE(I,J)=EE(I,J)+A(I,K)*E(K,J)                                     NAHS0319
C                                                                       NAHS0320
      WRITE OUTPUT TAPE KOUT, 2014                                      NAHS0321
      WRITE OUTPUT TAPE KOUT,2010,(TT(I),(EE(I,J),J=1,9),I=1,19)        NAHS0322
C                                                                       NAHS0323
```

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C     DETERMINE FOURIER COEFFICIENTS AS A FUNCTION OF SPACE          NAHS0324
C                                                                    NAHS0325
      XL=0.0                                                         NAHS0326
      DO 300 I=1,9                                                   NAHS0327
      A(I,1)=1.0                                                     NAHS0328
      A(I,2)=XL                                                      NAHS0329
      A(I,3)=XL**2                                                   NAHS0330
      A(I,4)=A(I,3)*XL                                               NAHS0331
      A(I,5)=A(I,4)*XL                                               NAHS0332
      A(I,6)=A(I,5)*XL                                               NAHS0333
      XL=XL+SQRTF((RR(I+1)-RR(I))**2+(ZZ(I+1)-ZZ(I))**2)             NAHS0334
      DO 300 J=1,NH                                                  NAHS0335
  300 EE(I,J)=E(J,I)                                                 NAHS0336
C                                                                    NAHS0337
      CALL LEAST(A,EE,B,D,E,9,6,NH)                                  NAHS0338
C                                                                    NAHS0339
      RETURN                                                         NAHS0340
C                                                                    NAHS0341
C     FORMAT STATEMENTS                                              NAHS0342
C                                                                    NAHS0343
 1000 FORMAT (12A6/ 6I5,2F10.2/ 4F10.2/4F10.2)                       NAHS0344
 1001 FORMAT (4F10.2)                                                NAHS0345
 1002 FORMAT (4F10.2)                                                NAHS0346
 1003 FORMAT (9F8.0)                                                 NAHS0347
 2000 FORMAT ( 1H1 12A6/                                             NAHS0348
     1 33H0NUMBER OF POINTS ALONG LENGTH--- I3/                      NAHS0349
     2 33H NUMBER OF POINTS THRU THICKNESS- I3/                      NAHS0350
     3 33H LOCATION OF BOND LINE----------- I3/                      NAHS0351
     4 33H NUMBER OF PROPERTY CARDS-------- I3/                      NAHS0352
     5 33H NUMBER OF HARMONICS------------- I4/                      NAHS0353
     6 33H NUMBER OF BOUNDARY CONDITIONS--- I4/                      NAHS0354
     7 33H SURFACE TEMPERATURE OF ABLATOR-- F6.0/                    NAHS0355
     8 33H ZERO STRESS TEMPERATURE--------- F6.0)                    NAHS0356
 2001 FORMAT (15H1    R-ORDINATE 5X 10HZ-ORDINATE 5X 10HBOND TEMP. 3X NAHS0357
     1 17HABLATOR THICKNESS / (3F15.3,1F20.4))                       NAHS0358
 2002 FORMAT (15H0  TEMPERATURE 5X 10HMODULUS A 5X 9HMODULUS B   1X  NAHS0359
     120H COEFF. OF EXPANSION /(3F15.0,1E20.5))                      NAHS0360
 2003 FORMAT (14H1  R-ORDINATES )                                    NAHS0361
 2004 FORMAT (14H1  Z-ORDINATES )                                    NAHS0362
 2005 FORMAT (14H1  TEMPERATURE )                                    NAHS0363
 2006 FORMAT (37H0PROPERTIES OF SANDWICH CORE MATERIAL )             NAHS0364
 2007 FORMAT (30H0PROPERTIES OF SANDWICH PLATES )                    NAHS0365
 2008 FORMAT                                                         NAHS0366
     1(28H  MODULUS OF ELASTICITY----- F10.0/                        NAHS0367
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    2 28H  POISSONS RATIO------------ F10.4/                             NAHS0368
    3 28H  COEFFICIENT OF EXPANSION-- F10.8/                             NAHS0369
    4 28H  THICKNESS----------------- F10.4)                             NAHS0370
 2009 FORMAT ( 5H1  R=9F12.4/5H   Z=9F12.4)                              NAHS0371
 2010 FORMAT (/(F5.0,9F12.1))                                            NAHS0372
 2011 FORMAT (20H1MATERIAL PROPERTIES)                                   NAHS0373
 2012 FORMAT (38H1LEAST SQUARE EVALUATION OF ABOVE DATA )                NAHS0374
 2013 FORMAT (36H0ANGLE        BOND LINE TEMPERATURES )                  NAHS0375
 2014 FORMAT (1H010X,47H FOURIER SERIES EXPANSION OF ABOVE TEMPERATURES)NAHS0376
      END                                                                NAHS0377
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      SUBROUTINE STIFFT(IX,JX,KX)                                       NAHS0378
C                                                                       NAHS0379
      DIMENSION R(30,8),Z(30,8),T(30,8),TC(30,8),E(10,10),CC(10,10),    NAHS0380
     1 XX(5),YY(5),S(15,15),P(15),G(9,9),B(9,9),C(9,9),LM(3),EE(9)      NAHS0381
      COMMON KINN,KOUT,N,M,L,NMAX,MMAX,MB,NMT,NH,NB,TE,TTT,ES,XS,AS,TS, NAHS0382
     1 ESH,XSH,ASH,TSH,R,Z,T,TC,E,CC,XX,YY,S,P,C,B,G                    NAHS0383
C                                                                       NAHS0384
C     INITIALIZATION                                                    NAHS0385
C                                                                       NAHS0386
      II=IX                                                             NAHS0387
      JJ=JX                                                             NAHS0388
      KK=KX                                                             NAHS0389
      DO 50 I=1,81                                                      NAHS0390
      G(I)=0.0                                                          NAHS0391
   50 B(I)=0.0                                                          NAHS0392
      AJ=XX(JJ)-XX(II)                                                  NAHS0393
      AK=XX(KK)-XX(II)                                                  NAHS0394
      BJ=YY(JJ)-YY(II)                                                  NAHS0395
      BK=YY(KK)-YY(II)                                                  NAHS0396
      AREA=(AJ*BK-BJ*AK)/2.                                             NAHS0397
      RBAR=(XX(II)+XX(JJ)+XX(KK))/3.                                    NAHS0398
C                                                                       NAHS0399
C     CALCULATE MATERIAL PROPERTIES                                     NAHS0400
C                                                                       NAHS0401
      TT=(T(N,M)+T(N,M+1)+T(N+1,M)+T(N+1,M+1))/4.                       NAHS0402
      DO 60 I=1,3                                                       NAHS0403
   60 EE(I)=CC(1,I)+TT*(CC(2,I)+TT*(CC(3,I)+TT*(CC(4,I)+TT*CC(5,I))))   NAHS0404
      IF(MB-M) 70,70,80                                                 NAHS0405
   70 EP=EE(1)                                                          NAHS0406
      ALP=EE(3)                                                         NAHS0407
      XNU=0.3155+0.000174*TT                                            NAHS0408
      GO TO 90                                                          NAHS0409
   80 EP=ES                                                             NAHS0410
      ALP=AS                                                            NAHS0411
      XNU=XS                                                            NAHS0412
   90 CONTINUE                                                          NAHS0413
C                                                                       NAHS0414
C     FORM MATRIX (B)                                                   NAHS0415
C                                                                       NAHS0416
      COMM=EP/((1.+XNU)*(1.-2.*XNU))                                    NAHS0417
      CX=(.5-XNU)*COMM                                                  NAHS0418
      DX=(1.-XNU)*COMM                                                  NAHS0419
      EX=XNU*COMM                                                       NAHS0420
      B(1,1)=DX                                                         NAHS0421
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      B(1,2)=EX                                                         NAHS0422
      B(1,3)=EX                                                         NAHS0423
      B(2,1)=EX                                                         NAHS0424
      B(2,2)=DX                                                         NAHS0425
      B(2,3)=EX                                                         NAHS0426
      B(3,1)=EX                                                         NAHS0427
      B(3,2)=EX                                                         NAHS0428
      B(3,3)=DX                                                         NAHS0429
      B(4,4)=CX                                                         NAHS0430
      B(5,5)=CX                                                         NAHS0431
      B(6,6)=CX                                                         NAHS0432
C                                                                       NAHS0433
C      FORM  G MATRIX                                                   NAHS0434
C                                                                       NAHS0435
      D=XX(JJ)*(YY(KK)-YY(II))+XX(II)*(YY(JJ)-YY(KK))                   NAHS0436
     1 +XX(KK)*(YY(II)-YY(JJ))                                          NAHS0437
      XN=L-1                                                            NAHS0438
C                                                                       NAHS0439
      G(1,1)=(YY(JJ)-YY(KK))/D                                          NAHS0440
      G(1,4)=(YY(KK)-YY(II))/D                                          NAHS0441
      G(1,7)=(YY(II)-YY(JJ))/D                                          NAHS0442
      G(2,1)=1./(XX(II)+XX(JJ)+XX(KK))                                  NAHS0443
      G(2,4)=G(2,1)                                                     NAHS0444
      G(2,7)=G(2,1)                                                     NAHS0445
      G(2,2)=XN*G(2,1)                                                  NAHS0446
      G(2,5)=XN*G(2,4)                                                  NAHS0447
      G(2,8)=XN*G(2,7)                                                  NAHS0448
      G(3,3)=(XX(KK)-XX(JJ))/D                                          NAHS0449
      G(3,6)=(XX(II)-XX(KK))/D                                          NAHS0450
      G(3,9)=(XX(JJ)-XX(II))/D                                          NAHS0451
      G(4,1)=-G(2,2)                                                    NAHS0452
      G(4,4)=-G(2,5)                                                    NAHS0453
      G(4,7)=-G(2,8)                                                    NAHS0454
      G(4,2)=G(1,1)-G(2,1)                                              NAHS0455
      G(4,5)=G(1,4)-G(2,4)                                              NAHS0456
      G(4,8)=G(1,7)-G(2,7)                                              NAHS0457
      G(5,1)=G(3,3)                                                     NAHS0458
      G(5,4)=G(3,6)                                                     NAHS0459
      G(5,7)=G(3,9)                                                     NAHS0460
      G(5,3)=G(1,1)                                                     NAHS0461
      G(5,6)=G(1,4)                                                     NAHS0462
      G(5,9)=G(1,7)                                                     NAHS0463
      G(6,2)=G(3,3)                                                     NAHS0464
      G(6,5)=G(3,6)                                                     NAHS0465
```

```
      G(6,8)=G(3,9)                                                     NAHS0466
      G(6,3)=G(4,1)                                                     NAHS0467
      G(6,6)=G(4,4)                                                     NAHS0468
      G(6,9)=G(4,7)                                                     NAHS0469
C                                                                       NAHS0470
C     FORM RING STIFFNESS MATRIX (A)T(B)(A)                             NAHS0471
C                                                                       NAHS0472
      DO 150 I=1,6                                                      NAHS0473
      DO 150 J=1,9                                                      NAHS0474
      C(I,J)=0.0                                                        NAHS0475
      DO 150 K=1,6                                                      NAHS0476
  150 C(I,J)=C(I,J)+B(I,K)*G(K,J)                                       NAHS0477
C                                                                       NAHS0478
      COMM=RBAR*AREA                                                    NAHS0479
      DO 165 I=1,9                                                      NAHS0480
      DO 165 J=1,9                                                      NAHS0481
      B(I,J)=0.0                                                        NAHS0482
      DO 160 K=1,6                                                      NAHS0483
  160 B(I,J)=B(I,J)+G(K,I)*C(K,J)                                       NAHS0484
  165 B(I,J)=B(I,J)*COMM                                                NAHS0485
C                                                                       NAHS0486
C     COMPUTE TEMPERATURE LOADS                                         NAHS0487
C                                                                       NAHS0488
      TEM=(TC(N,M)+TC(N,M+1)+TC(N+1,M)+TC(N+1,M+1))/4.                  NAHS0489
      TTT=TEM*EP*ALP/(1.-2.*XNU)                                        NAHS0490
      TEM=TTT*RBAR*AREA                                                 NAHS0491
      DO 170 I=1,9                                                      NAHS0492
  170 EE(I)=(G(1,I)+G(2,I)+G(3,I))*TEM                                  NAHS0493
C                                                                       NAHS0494
C     ADD TRIANGULAR RING STIFFNESS TO QUADRILATERAL RING STIFFNESS     NAHS0495
C                                                                       NAHS0496
      LM(1)=3*II-3                                                      NAHS0497
      LM(2)=3*JJ-3                                                      NAHS0498
      LM(3)=3*KK-3                                                      NAHS0499
      DO 250 I=1,3                                                      NAHS0500
      DO 250 K=1,3                                                      NAHS0501
      II=LM(I)+K                                                        NAHS0502
      KK=3*I-3+K                                                        NAHS0503
      P(II)=P(II)+EE(KK)                                                NAHS0504
      DO 250 J=1,3                                                      NAHS0505
      DO 250 LX=1,3                                                     NAHS0506
      JJ=LM(J)+LX                                                       NAHS0507
      LL=3*J-3+LX                                                       NAHS0508
  250 S(II,JJ)=S(II,JJ)+B(KK,LL)                                        NAHS0509
```

```
C                                                                       NAHS0510
      RETURN                                                            NAHS0511
C                                                                       NAHS0512
      END                                                               NAHS0513
```

```
      SUBROUTINE STIFFP                                                 NAHS0514
C                                                                       NAHS0515
      DIMENSION R(30,8),Z(30,8),T(30,8),TC(30,8),E(10,10),CC(10,10),    NAHS0516
     1 XX(5),YY(5),S(15,15),P(15),G(9,9),B(9,9),C(9,9)                  NAHS0517
      COMMON KINN,KOUT,N,M,L,NMAX,MMAX,MB,NMT,NH,NB,TE,TTT,ES,XS,AS,TS, NAHS0518
     1 ESH,XSH,ASH,TSH,R,Z,T,TC,E,CC,XX,YY,S,P,C,B,G                    NAHS0519
C                                                                       NAHS0520
C     FORMATION OF G MATRIX                                             NAHS0521
C                                                                       NAHS0522
      XN=L-1                                                            NAHS0523
      A=R(N+1,M)-R(N,M)                                                 NAHS0524
      B=Z(N+1,M)-Z(N,M)                                                 NAHS0525
      XL2=A**2+B**2                                                     NAHS0526
      XL=SQRTF(XL2)                                                     NAHS0527
      RBAR=R(N+1,M)+R(N,M)                                              NAHS0528
C                                                                       NAHS0529
      G(1,1)=-A/XL2                                                     NAHS0530
      G(1,2)=0.0                                                        NAHS0531
      G(1,3)=-B/XL2                                                     NAHS0532
      G(1,4)=-G(1,1)                                                    NAHS0533
      G(1,5)=0.0                                                        NAHS0534
      G(1,6)=-G(1,3)                                                    NAHS0535
      G(2,1)=1./RBAR                                                    NAHS0536
      G(2,2)=XN*G(2,1)                                                  NAHS0537
      G(2,3)=0.0                                                        NAHS0538
      G(2,4)=G(2,1)                                                     NAHS0539
      G(2,5)=G(2,2)                                                     NAHS0540
      G(2,6)=0.0                                                        NAHS0541
      G(3,1)=-XN*A/RBAR/XL                                              NAHS0542
      G(3,2)=1./XL                                                      NAHS0543
      G(3,3)=-XN*B/RBAR/XL                                              NAHS0544
      G(3,4)=G(3,1)                                                     NAHS0545
      G(3,5)=-G(3,2)                                                    NAHS0546
      G(3,6)=G(3,3)                                                     NAHS0547
C                                                                       NAHS0548
C     FORMATION OF B MATRIX                                             NAHS0549
C                                                                       NAHS0550
      B(1,1)=ESH/(1.-XSH**2)                                            NAHS0551
      B(1,2)=XSH*B(1,1)                                                 NAHS0552
      B(1,3)=0.0                                                        NAHS0553
      B(2,1)=B(1,2)                                                     NAHS0554
      B(2,2)=B(1,1)                                                     NAHS0555
      B(2,3)=0.0                                                        NAHS0556
      B(3,1)=0.0                                                        NAHS0557
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      B(3,2)=0.0                                                        NAHS0558
      B(3,3)=.5*ESH/(1.+XSH)                                            NAHS0559
C                                                                       NAHS0560
C     FORMATION OF PLATE STIFFNESS MATRIX (G)T*(B)*(G)                  NAHS0561
C                                                                       NAHS0562
      DO 150 I=1,3                                                      NAHS0563
      DO 150 J=1,6                                                      NAHS0564
      C(I,J)=0.0                                                        NAHS0565
      DO 150 K=1,3                                                      NAHS0566
  150 C(I,J)=C(I,J)+B(I,K)*G(K,J)                                       NAHS0567
C                                                                       NAHS0568
      COMM=RBAR*XL*TSH/2.                                               NAHS0569
      DO 165 I=1,6                                                      NAHS0570
      DO 165 J=1,6                                                      NAHS0571
      S(I,J)=0.0                                                        NAHS0572
      DO 160 K=1,3                                                      NAHS0573
  160 S(I,J)=S(I,J)+G(K,I)*C(K,J)                                       NAHS0574
  165 S(I,J)=S(I,J)*COMM                                                NAHS0575
C                                                                       NAHS0576
C     COMPUTE TEMPERATURE LOADS                                         NAHS0577
C                                                                       NAHS0578
      TEM=(TC(N,M)+TC(N+1,M))/2.                                        NAHS0579
      TTT=TEM*ESH*ASH*(1.+XSH)/(1.-XSH**2)                              NAHS0580
      TEM=TTT*COMM                                                      NAHS0581
      DO 170 I=1,6                                                      NAHS0582
  170 P(I)=(G(1,I)+G(2,I))*TEM                                          NAHS0583
C                                                                       NAHS0584
      RETURN                                                            NAHS0585
C                                                                       NAHS0586
      END                                                               NAHS0587
```

```
      SUBROUTINE STRESS                                                 NAHS0588
      DIMENSION R(30,8),Z(30,8),T(30,8),TC(30,8),E(10,10),CC(10,10),    NAHS0589
     1 XX(5),YY(5),S(15,15),P(15),U(30,8),V(30,8),W(30,8),KODE(30,10),  NAHS0590
     2 C(9,9),RR(720),SIG(6,30,8),PSIG(3,30,2),B(9,9),G(9,9)            NAHS0591
      COMMON KINN,KOUT,N,M,L,NMAX,MMAX,MB,NMT,NH,NB,TE,TTT,ES,XS,AS,TS, NAHS0592
     1 ESH,XSH,ASH,TSH,R,Z,T,TC,E,CC,XX,YY,S,P,C,B,G,KODE,RR,U,V,W,SIG, NAHS0593
     3 PSIG                                                             NAHS0594
C                                                                       NAHS0595
C     SEPARATE DISPLACEMENTS FOR HARMONIC L                             NAHS0596
C                                                                       NAHS0597
      K=1                                                               NAHS0598
      DO 400 N=1,NMAX                                                   NAHS0599
      DO 400 M=1,MMAX                                                   NAHS0600
      U(N,M)=RR(K)                                                      NAHS0601
      V(N,M)=RR(K+1)                                                    NAHS0602
      W(N,M)=RR(K+2)                                                    NAHS0603
  400 K=K+3                                                             NAHS0604
C                                                                       NAHS0605
C     CALCULATION OF STRESSES FOR HARMONIC L                            NAHS0606
C                                                                       NAHS0607
      NN=NMAX-1                                                         NAHS0608
      MM=MMAX-1                                                         NAHS0609
      DO 100 N=1,NN                                                     NAHS0610
      N=N                                                               NAHS0611
      DO 100 M=1,MM                                                     NAHS0612
      M=M                                                               NAHS0613
      XX(1)=R(N,M)                                                      NAHS0614
      XX(2)=R(N,M+1)                                                    NAHS0615
      XX(3)=R(N+1,M)                                                    NAHS0616
      XX(4)=R(N+1,M+1)                                                  NAHS0617
      YY(1)=Z(N,M)                                                      NAHS0618
      YY(2)=Z(N,M+1)                                                    NAHS0619
      YY(3)=Z(N+1,M)                                                    NAHS0620
      YY(4)=Z(N+1,M+1)                                                  NAHS0621
C                                                                       NAHS0622
      CALL STIFFT(1,4,2)                                                NAHS0623
C                                                                       NAHS0624
      P(1)=U(N,M)                                                       NAHS0625
      P(2)=V(N,M)                                                       NAHS0626
      P(3)=W(N,M)                                                       NAHS0627
      P(4)=U(N+1,M+1)                                                   NAHS0628
      P(5)=V(N+1,M+1)                                                   NAHS0629
      P(6)=W(N+1,M+1)                                                   NAHS0630
      P(7)=U(N,M+1)                                                     NAHS0631
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      P(8)=V(N,M+1)                                                     NAHS0632
      P(9)=W(N,M+1)                                                     NAHS0633
C                                                                       NAHS0634
      DO 80 I=1,6                                                       NAHS0635
      SIG(I,N,M)=0.0                                                    NAHS0636
      DO 80 K=1,9                                                       NAHS0637
   80 SIG(I,N,M)=SIG(I,N,M)+C(I,K)*P(K)                                 NAHS0638
C                                                                       NAHS0639
      CALL STIFFT(1,3,4)                                                NAHS0640
C                                                                       NAHS0641
      P(1)=U(N,M)                                                       NAHS0642
      P(2)=V(N,M)                                                       NAHS0643
      P(3)=W(N,M)                                                       NAHS0644
      P(4)=U(N+1,M)                                                     NAHS0645
      P(5)=V(N+1,M)                                                     NAHS0646
      P(6)=W(N+1,M)                                                     NAHS0647
      P(7)=U(N+1,M+1)                                                   NAHS0648
      P(8)=V(N+1,M+1)                                                   NAHS0649
      P(9)=W(N+1,M+1)                                                   NAHS0650
C                                                                       NAHS0651
      DO 95 I=1,6                                                       NAHS0652
      DO 90 K=1,9                                                       NAHS0653
   90 SIG(I,N,M)=SIG(I,N,M)+C(I,K)*P(K)                                 NAHS0654
   95 SIG(I,N,M)=SIG(I,N,M)/2.                                          NAHS0655
      SIG(1,N,M)=SIG(1,N,M)-TTT                                         NAHS0656
      SIG(2,N,M)=SIG(2,N,M)-TTT                                         NAHS0657
  100 SIG(3,N,M)=SIG(3,N,M)-TTT                                         NAHS0658
C                                                                       NAHS0659
      WRITE TAPE 20,((U(N,M),W(N,M),V(N,M),(SIG(I,N,M),I=1,6),          NAHS0660
     1 N=1,NMAX),M=1,MMAX)                                              NAHS0661
C                                                                       NAHS0662
C     COMPUTE STRESSES IN FACE PLATES                                   NAHS0663
C                                                                       NAHS0664
      DO 210 N=1,NN                                                     NAHS0665
      N=N                                                               NAHS0666
      DO 210 K=1,2                                                      NAHS0667
      M=(K-1)*(MB-1)+1                                                  NAHS0668
      CALL STIFFP                                                       NAHS0669
      P(1)=U(N,M)                                                       NAHS0670
      P(2)=V(N,M)                                                       NAHS0671
      P(3)=W(N,M)                                                       NAHS0672
      P(4)=U(N+1,M)                                                     NAHS0673
      P(5)=V(N+1,M)                                                     NAHS0674
      P(6)=W(N+1,M)                                                     NAHS0675
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```
      DO 200 I=1,3                                                      NAHS0676
      PSIG(I,N,K)=0.0                                                   NAHS0677
      DO 200 J=1,6                                                      NAHS0678
  200 PSIG(I,N,K)=PSIG(I,N,K)+C(I,J)*P(J)                               NAHS0679
      PSIG(1,N,K)=PSIG(1,N,K)-TTT                                       NAHS0680
  210 PSIG(2,N,K)=PSIG(2,N,K)-TTT                                       NAHS0681
C                                                                       NAHS0682
      WRITE TAPE 20, ((( PSIG(I,N,K),I=1,3),K=1,2),N=1,NN)              NAHS0683
C                                                                       NAHS0684
      RETURN                                                            NAHS0685
C                                                                       NAHS0686
      END                                                               NAHS0687
```

```
      SUBROUTINE OUTPUT                                                 NAHS0688
C                                                                       NAHS0689
      DIMENSION U(30,8),V(30,8),W(30,8),XU(30,8),XV(30,8),XW(30,8),     NAHS0690
     1 SIG(6,30,8),XSIG(6,30,8),PSIG(3,30,2),XPSIG(3,30,2)              NAHS0691
      COMMON KINN,KOUT,N,M,L,NMAX,MMAX,MB,NMT,NH,NB,TE,TTT,ES,XS,AS,TS, NAHS0692
     1 U,V,W,XU,XV,XW,SIG,XSIG,PSIG,XPSIG                               NAHS0693
C                                                                       NAHS0694
      NN=NMAX-1                                                         NAHS0695
      MM=MMAX-1                                                         NAHS0696
C                                                                       NAHS0697
   50 READ INPUT TAPE KINN,1000,THETA                                   NAHS0698
C                                                                       NAHS0699
      DO 110 N=1,NMAX                                                   NAHS0700
      DO 100 M=1,MMAX                                                   NAHS0701
      U(N,M)=0.0                                                        NAHS0702
      V(N,M)=0.0                                                        NAHS0703
      W(N,M)=0.0                                                        NAHS0704
      DO 100 I=1,6                                                      NAHS0705
  100 SIG(I,N,M)=0.0                                                    NAHS0706
      DO 110 K=1,2                                                      NAHS0707
      DO 110 I=1,3                                                      NAHS0708
  110 PSIG(I,N,K)=0.0                                                   NAHS0709
C                                                                       NAHS0710
      REWIND 20                                                         NAHS0711
      DO 500 LL=1,NH                                                    NAHS0712
      TT=FLOATF(LL-1)*THETA*.017453289                                  NAHS0713
      COSINE=COSF(TT)                                                   NAHS0714
      SINE=SINF(TT)                                                     NAHS0715
C                                                                       NAHS0716
      READ  TAPE 20,((XU(N,M),XW(N,M),XV(N,M),(XSIG(I,N,M),I=1,6),      NAHS0717
     1 N=1,NMAX),M=1,MMAX)                                              NAHS0718
C                                                                       NAHS0719
      DO 200 N=1,NMAX                                                   NAHS0720
      DO 200 M=1,MMAX                                                   NAHS0721
      U(N,M)=U(N,M)+XU(N,M)*COSINE                                      NAHS0722
      W(N,M)=W(N,M)+XW(N,M)*COSINE                                      NAHS0723
      V(N,M)=V(N,M)+XV(N,M)*SINE                                        NAHS0724
      DO 150 I=1,3                                                      NAHS0725
  150 SIG(I,N,M)=SIG(I,N,M)+XSIG(I,N,M)*COSINE                          NAHS0726
      SIG(4,N,M)=SIG(4,N,M)+XSIG(4,N,M)*SINE                            NAHS0727
      SIG(5,N,M)=SIG(5,N,M)+XSIG(5,N,M)*COSINE                          NAHS0728
  200 SIG(6,N,M)=SIG(6,N,M)+XSIG(6,N,M)*SINE                            NAHS0729
C                                                                       NAHS0730
      READ TAPE 20,  (((XPSIG(I,N,K),I=1,3),K=1,2),N=1,NN)              NAHS0731
```

```
C                                                                       NAHS0732
      DO 300 N=1,NN                                                     NAHS0733
      DO 300 K=1,2                                                      NAHS0734
      PSIG(1,N,K)=PSIG(1,N,K)+XPSIG(1,N,K)*COSINE                       NAHS0735
      PSIG(2,N,K)=PSIG(2,N,K)+XPSIG(2,N,K)*COSINE                       NAHS0736
  300 PSIG(3,N,K)=PSIG(3,N,K)+XPSIG(3,N,K)*SINE                         NAHS0737
C                                                                       NAHS0738
  500 CONTINUE                                                          NAHS0739
C                                                                       NAHS0740
C     OUTPUT DISPLACEMENTS AND STRESSES                                 NAHS0741
C                                                                       NAHS0742
      WRITE OUTPUT TAPE KOUT,2003, THETA                                NAHS0743
      CALL PRINTM(U,NMAX,MMAX,30)                                       NAHS0744
      WRITE OUTPUT TAPE KOUT,2004, THETA                                NAHS0745
      CALL PRINTM(W,NMAX,MMAX,30)                                       NAHS0746
      WRITE OUTPUT TAPE KOUT,2005, THETA                                NAHS0747
      CALL PRINTM(V,NMAX,MMAX,30)                                       NAHS0748
C                                                                       NAHS0749
      DO 600 M=1,MM                                                     NAHS0750
      WRITE OUTPUT TAPE KOUT, 2007, THETA                               NAHS0751
  600 WRITE OUTPUT TAPE KOUT,2000,  (N,M,(SIG(I,N,M),I=1,6),N=1,NN)     NAHS0752
C                                                                       NAHS0753
      WRITE OUTPUT TAPE KOUT, 2008, THETA                               NAHS0754
      WRITE OUTPUT TAPE KOUT,2006,(N,((PSIG(I,N,K),I=1,3),K=1,2),N=1,NN)NAHS0755
C                                                                       NAHS0756
      GO TO 50                                                          NAHS0757
C                                                                       NAHS0758
 1000 FORMAT (F5.0)                                                     NAHS0759
 2000 FORMAT (12H0    N     M 10X,2HRR 10X,2HTT 10X,2HZZ 10X,2HRT       NAHS0760
     1 10X,2HRZ 10X,2HTZ / (2I6,6F12.2))                                NAHS0761
 2003 FORMAT (20H1R-DISPLACEMENTS AT  F4.0,8H DEGREES)                  NAHS0762
 2004 FORMAT (20H1Z-DISPLACEMENTS AT  F4.0,8H DEGREES)                  NAHS0763
 2005 FORMAT (20H1T-DISPLACEMENTS AT  F4.0,8H DEGREES)                  NAHS0764
 2006 FORMAT (I5,6F18.1)                                                NAHS0765
 2007 FORMAT  (28H1AVERAGE ELEMENT STRESSES AT F5.0,8H DEGREES)         NAHS0766
 2008 FORMAT (44H1AVERAGE STRESSES IN SANDWICH FACE PLATES ATF5.0,8H DEGNAHS0767
     1REES//22X, 12HBOTTOM PLATE 44X, 9HTOP PLATE /5H    N 2(54H     MERNAHS0768
     1ID. STRESS       HOOP STRESS    IN-PLANE SHEAR ))                 NAHS0769
C                                                                       NAHS0770
      END                                                               NAHS0771
```

```
      SUBROUTINE LEAST(A,B,C,D,E,N,M,L)                                 NAHS0772
C                                                                       NAHS0773
      DIMENSION A(50,10),B(50,10),C(10,10),D(10,10),E(10,10)            NAHS0774
C                                                                       NAHS0775
C     (C)=(A)T*(A)  AND (D)=(A)T*(B)                                    NAHS0776
C                                                                       NAHS0777
      DO 200 I=1,M                                                      NAHS0778
      DO 100 J=1,M                                                      NAHS0779
      C(I,J)=0.0                                                        NAHS0780
      DO 100 K=1,N                                                      NAHS0781
  100 C(I,J)=C(I,J)+A(K,I)*A(K,J)                                       NAHS0782
      DO 200 J=1,L                                                      NAHS0783
      D(I,J)=0.0                                                        NAHS0784
      DO 200 K=1,N                                                      NAHS0785
  200 D(I,J)=D(I,J)+A(K,I)*B(K,J)                                       NAHS0786
C                                                                       NAHS0787
C     INVERT (C)                                                        NAHS0788
C                                                                       NAHS0789
      CALL INVERT(C,M,10,B,E)                                           NAHS0790
C                                                                       NAHS0791
C      (E)=(C)*(D)                                                      NAHS0792
C                                                                       NAHS0793
      DO 300 I=1,M                                                      NAHS0794
      DO 300 J=1,L                                                      NAHS0795
      E(I,J)=0.0                                                        NAHS0796
      DO 300 K=1,M                                                      NAHS0797
  300 E(I,J)=E(I,J)+C(I,K)*D(K,J)                                       NAHS0798
C                                                                       NAHS0799
C     CHECK RESULTS  (B)=(A)*(E)                                        NAHS0800
C                                                                       NAHS0801
      DO 400 I=1,N                                                      NAHS0802
      DO 400 J=1,L                                                      NAHS0803
      B(I,J)=0.0                                                        NAHS0804
      DO 400 K=1,M                                                      NAHS0805
  400 B(I,J)=B(I,J)+A(I,K)*E(K,J)                                       NAHS0806
C                                                                       NAHS0807
      RETURN                                                            NAHS0808
C                                                                       NAHS0809
      END                                                               NAHS0810
```

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      SUBROUTINE INVERT(A,NN,N,M,C)                                     NAHS0811
C                                                                       NAHS0812
C           GENERAL MATRIX INVERSION SUBROUTINE                         NAHS0813
C                                                                       NAHS0814
      DIMENSION A(1),M(1),C(1)                                          NAHS0815
C                                                                       NAHS0816
C     CHECK FOR 1 X 1 MATRIX                                            NAHS0817
C                                                                       NAHS0818
      IF(NN-1) 300,70,80                                                NAHS0819
   70 A(1)=1./A(1)                                                      NAHS0820
      GO TO 300                                                         NAHS0821
C                                                                       NAHS0822
   80 DO 90 I=1,NN                                                      NAHS0823
   90 M(I)=-I                                                           NAHS0824
C                                                                       NAHS0825
      DO 140 I=1,NN                                                     NAHS0826
C                                                                       NAHS0827
C     LOCATE LARGEST ELEMENT                                            NAHS0828
C                                                                       NAHS0829
      D=0.0                                                             NAHS0830
      DO 112 L=1,NN                                                     NAHS0831
      IF (M(L)) 100,100,112                                             NAHS0832
  100 J=L                                                               NAHS0833
      DO 110 K=1,NN                                                     NAHS0834
      IF (M(K)) 103,103,108                                             NAHS0835
  103 IF (ABSF(D)-ABSF(A(J))) 105,105,108                               NAHS0836
  105 LD=L                                                              NAHS0837
      KD=K                                                              NAHS0838
      D=A(J)                                                            NAHS0839
  108 J=J+N                                                             NAHS0840
  110 CONTINUE                                                          NAHS0841
  112 CONTINUE                                                          NAHS0842
C                                                                       NAHS0843
C     INTERCHANGE ROWS                                                  NAHS0844
C                                                                       NAHS0845
      TEMP=-M(LD)                                                       NAHS0846
      M(LD)=M(KD)                                                       NAHS0847
      M(KD)=TEMP                                                        NAHS0848
      L=LD                                                              NAHS0849
      K=KD                                                              NAHS0850
      DO 114 J=1,NN                                                     NAHS0851
      C(J)=A(L)                                                         NAHS0852
      A(L)=A(K)                                                         NAHS0853
      A(K)=C(J)                                                         NAHS0854
```

```
      L=L+N                                                             NAHS0855
  114 K=K+N                                                             NAHS0856
C                                                                       NAHS0857
C     DIVIDE COLUNM BY LARGEST ELEMENT                                  NAHS0858
C                                                                       NAHS0859
      NR=(KD-1)*N+1                                                     NAHS0860
      NH=NR+N-1                                                         NAHS0861
      DO 115 K=NR,NH                                                    NAHS0862
  115 A(K)=A(K)/D                                                       NAHS0863
C                                                                       NAHS0864
C     REDUCE REMAINING ROWS AND COLUMNS                                 NAHS0865
C                                                                       NAHS0866
      L=1                                                               NAHS0867
      DO 135 J=1,NN                                                     NAHS0868
      IF (J-KD) 130,125,130                                             NAHS0869
  125 L=L+N                                                             NAHS0870
      GO TO 135                                                         NAHS0871
  130 DO 134 K=NR,NH                                                    NAHS0872
      A(L)=A(L)-C(J)*A(K)                                               NAHS0873
  134 L=L+1                                                             NAHS0874
  135 CONTINUE                                                          NAHS0875
C                                                                       NAHS0876
C     REDUCE ROW                                                        NAHS0877
C                                                                       NAHS0878
      C(KD)=-1.0                                                        NAHS0879
      J=KD                                                              NAHS0880
      DO 140 K=1,NN                                                     NAHS0881
      A(J)=-C(K)/D                                                      NAHS0882
  140 J=J+N                                                             NAHS0883
C                                                                       NAHS0884
C     INTERCHANGE COLUMNS                                               NAHS0885
C                                                                       NAHS0886
      DO 200 I=1,NN                                                     NAHS0887
      L=0                                                               NAHS0888
  150 L=L+1                                                             NAHS0889
      IF(M(L)-I) 150,160,150                                            NAHS0890
  160 K=(L-1)*N+1                                                       NAHS0891
      J=(I-1)*N+1                                                       NAHS0892
      M(L)=M(I)                                                         NAHS0893
      M(I)=I                                                            NAHS0894
      DO 200 L=1,NN                                                     NAHS0895
      TEMP=A(K)                                                         NAHS0896
      A(K)=A(J)                                                         NAHS0897
      A(J)=TEMP                                                         NAHS0898
```

```
      J=J+1                                                    NAHS0899
  200 K=K+1                                                    NAHS0900
C                                                              NAHS0901
  300 RETURN                                                   NAHS0902
C                                                              NAHS0903
      END                                                      NAHS0904
```

```
      SUBROUTINE SYMSOL (A,B,NN,MM)                                     NAHS0905
C                                                                       NAHS0906
      DIMENSION A(720,30),B(720),C(30)                                  NAHS0907
C                                                                       NAHS0908
      N = 0                                                             NAHS0909
  100 N = N+1                                                           NAHS0910
C                                                                       NAHS0911
C     REDUCE N TH EQUATION                                              NAHS0912
C                                                                       NAHS0913
C        1. DIVIDE RIGHT SIDE BY DIAGONAL ELEMENT                       NAHS0914
C                                                                       NAHS0915
      B(N) = B(N) / A(N,1)                                              NAHS0916
C                                                                       NAHS0917
C        2. CHECK FOR LAST EQUATION                                     NAHS0918
C                                                                       NAHS0919
      IF(N-NN) 150,300,150                                              NAHS0920
C                                                                       NAHS0921
C        3. DIVIDE N TH EQUATION BY DIAGONAL ELEMENT                    NAHS0922
C                                                                       NAHS0923
  150 DO 200 K=2,MM                                                     NAHS0924
      C(K) = A(N,K)                                                     NAHS0925
  200 A(N,K) = A(N,K) / A(N,1)                                          NAHS0926
C                                                                       NAHS0927
C        4. REDUCE REMAINING EQUATIONS                                  NAHS0928
C                                                                       NAHS0929
      DO 260 L=2,MM                                                     NAHS0930
      I = N+L-1                                                         NAHS0931
      IF(NN-I) 260,240,240                                              NAHS0932
  240 J=0                                                               NAHS0933
      DO 250 K=L,MM                                                     NAHS0934
      J=J+1                                                             NAHS0935
  250 A(I,J) = A(I,J) - C(L) * A(N,K)                                   NAHS0936
      B(I) = B(I) - C(L) * B(N)                                         NAHS0937
  260 CONTINUE                                                          NAHS0938
      GO TO 100                                                         NAHS0939
C                                                                       NAHS0940
C     BACK SUBSTITUTION                                                 NAHS0941
C                                                                       NAHS0942
  300 N = N-1                                                           NAHS0943
C                                                                       NAHS0944
C        1. CHECK FOR FIRST EQUATION                                    NAHS0945
C                                                                       NAHS0946
      IF(N) 350,500,350                                                 NAHS0947
C                                                                       NAHS0948
```

```
C         2. CALCULATE UNKNOWN B(N)                                    NAHS0949
C                                                                      NAHS0950
  350 DO 400 K=2,MM                                                    NAHS0951
      L = N+K-1                                                        NAHS0952
      IF(NN-L) 400,370,370                                             NAHS0953
  370 B(N) = B(N) - A(N,K) * B(L)                                      NAHS0954
  400 CONTINUE                                                         NAHS0955
      GO TO 300                                                        NAHS0956
C                                                                      NAHS0957
  500 RETURN                                                           NAHS0958
C                                                                      NAHS0959
      END                                                              NAHS0960
```

```
      SUBROUTINE PRINTM (A,NR,NC,MAXR)                                  NAHS0961
      DIMENSION A(1),NHED(10)                                           NAHS0962
      COMMON KINN,KOUT                                                  NAHS0963
C                                                                       NAHS0964
      DO 50 I=1,NC,10                                                   NAHS0965
      II=NC-I+1                                                         NAHS0966
      IF (II-10) 20,20,10                                               NAHS0967
   10 II=10                                                             NAHS0968
   20 DO 30 J=1,II                                                      NAHS0969
   30 NHED(J)=I+J-1                                                     NAHS0970
C                                                                       NAHS0971
      WRITE OUTPUT TAPE KOUT,120,(NHED(J),J=1,II)                       NAHS0972
C                                                                       NAHS0973
      DO 50 J=1,NR                                                      NAHS0974
      KL=J+(I-1)*MAXR                                                   NAHS0975
      KH=KL+(II-1)*MAXR                                                 NAHS0976
   50 WRITE OUTPUT TAPE KOUT,130,(J,(A(K),K=KL,KH,MAXR))                NAHS0977
C                                                                       NAHS0978
      RETURN                                                            NAHS0979
C                                                                       NAHS0980
  120 FORMAT (8H0   N/M  10I11)                                         NAHS0981
  130 FORMAT (I5,3X,10F11.3)                                            NAHS0982
      END                                                               NAHS0983
```

APPENDIX G

SUMMARY OF EFFORTS TO SOLVE THE THERMAL STRAIN PROBLEM BY THE OVER-RELAXATION AND DIRECT INTEGRATION OF FINITE-DIFFERENCE FORMULATION

I. ACCOMPLISHMENTS

During the initial phases of this contract, finite-difference techniques were employed in an effort to solve the governing differential equations for this problem. The following work was accomplished:

A. Derivation of basic equations suitable to the given geometry

B. Derivation of the finite-differences model of the equations

C. Development of a thin-shell model for the thin layers (bond, face-plates) and its finite-differences equivalent

D. Attempt to establish proper boundary conditions for the finite-difference model

E. Programing of the finite-difference model(s) and attempts to solve the equations by (1) the over-relaxation approach (in several versions) and (2) direct matrix, inversions.

F. Investigation of alternative methods and recommendations.

A summary of the results in each of the above areas is presented in this appendix.

II. DERIVATION OF BASIC EQUATIONS SUITABLE TO GIVEN GEOMETRY

A complete set of equilibrium equations in terms of displacements in spherical and toroidal coordinates was derived and is given below. A singular point exists at $\phi = 0$. The equations for this point are also presented.

For orthogonal curvilinear coordinates $(\alpha_1, \alpha_2, \alpha_3)$, the element of arc ds is defined by

$$ds^2 = \sum_{i=1}^{3} g_{ii}\, d\alpha_i^2 \tag{G1}$$

where g_{ii} represents the metric coefficients

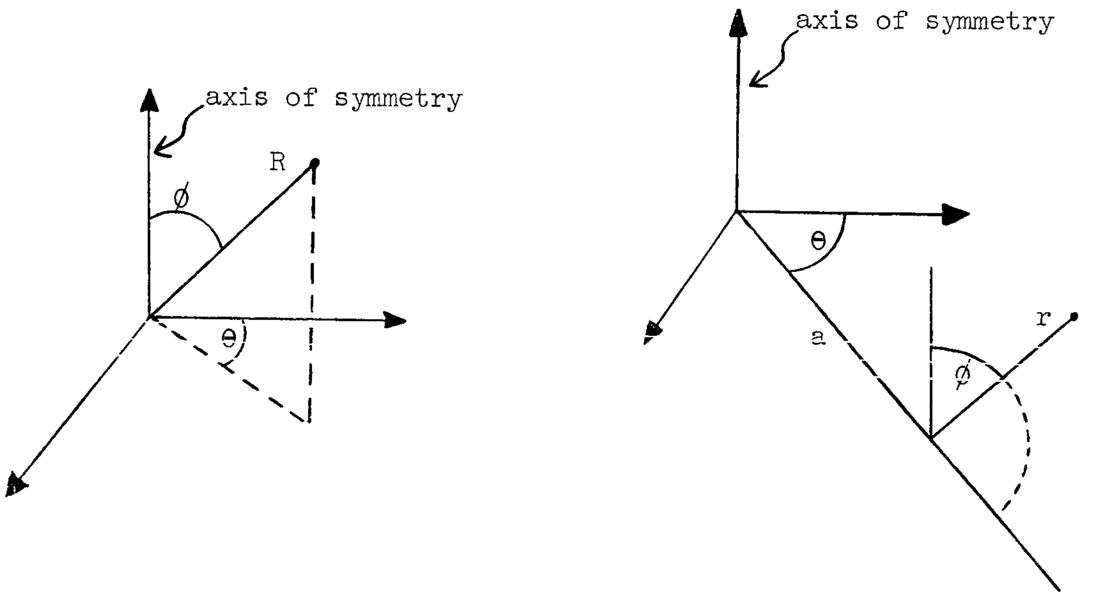


Fig. G1 - Coordinate Axes

	Spherical Coordinates	Toroidal Coordinates
α_1	R	r
α_2	ϕ	ϕ
α_3	θ	θ
g_{11}	1	1
g_{22}	R^2	r^2
g_{33}	$R^2 \sin^2 \phi$	$(a + r \sin \phi)^2$
$g \equiv \sqrt{g_{11} g_{22} g_{33}}$	$R^2 \sin \phi$	$r(a + r \sin \phi)$

Note: Toroidal coordinates reduce to spherical coordinates in the limit as $a \longrightarrow 0$.

Equations of equilibrium with zero body force take the form shown below.

$$\sum_{j=1}^{3}\left[\frac{\partial}{\partial\alpha_j}\left(\frac{g g_{ii}\tau_{ij}}{\sqrt{g_{ii}g_{jj}}}\right) - \frac{1}{2}\cdot\frac{g\tau_{jj}}{g_{jj}}\ \frac{\partial g_{jj}}{\partial\alpha_i}\right] = 0 \tag{G2}$$

where $g \equiv \sqrt{g_{11}g_{22}g_{33}}$ and τ_{ii} and τ_{ij} are normal and shear components of stress, respectively.

After substituting the respective components of α_i and g_{ii} in Equation (G2) and performing the indicated differentiations and summations, there are obtained the following equilibrium equations in terms of stresses for each coordinate system:

A. SPHERICAL COORDINATES

$$\frac{\partial\tau_{RR}}{\partial R} + \frac{1}{R}\ \frac{\partial\tau_{R\phi}}{\partial\phi} + \frac{1}{R\sin\phi}\ \frac{\partial\tau_{R\theta}}{\partial\theta} + \frac{2\tau_{RR} - \tau_{\phi\phi} - \tau_{\theta\theta} + \tau_{R\phi}\cot\phi}{R} = 0 \tag{G3}$$

$$\frac{\partial\tau_{R\phi}}{\partial R} + \frac{1}{R}\ \frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{1}{R\sin\phi}\ \frac{\partial\tau_{\phi\theta}}{\partial\theta} + \frac{3\tau_{R\phi} + (\tau_{\phi\phi} - \tau_{\theta\theta})\cot\phi}{R} = 0 \tag{G4}$$

$$\frac{\partial\tau_{R\theta}}{\partial R} + \frac{1}{R}\ \frac{\partial\tau_{\phi\theta}}{\partial\phi} + \frac{1}{R\sin\phi}\ \frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{3\tau_{R\theta} + 2\tau_{\phi\theta}\cot\phi}{R} = 0 \tag{G5}$$

B. TOROIDAL COORDINATES

$$\frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\ \frac{\partial\tau_{r\phi}}{\partial\phi} + \frac{1}{(a + r\sin\phi)}\ \frac{\partial\tau_{r\theta}}{\partial\theta}$$

$$+ \frac{(a + 2r\sin\phi)\tau_{rr} - (a + r\sin\phi)\tau_{\phi\phi} - \tau_{\theta\theta}\, r\sin\phi + \tau_{r\phi}\, r\cos\phi}{r(a + r\sin\phi)} = 0 \tag{G6}$$

$$\frac{\partial\tau_{r\phi}}{\partial r} + \frac{1}{r}\ \frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{1}{a + r\sin\phi}\ \frac{\partial\tau_{\phi\theta}}{\partial\theta}$$

$$+ \frac{(2a + 3r\sin\phi)\tau_{r\phi} + (\tau_{\phi\phi} - \tau_{\theta\theta})\, r\cos\phi}{r(a + r\sin\phi)} = 0 \tag{G7}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{1}{(a + r\sin\phi)}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{(a + 3r\sin\phi)\tau_{r\theta} + 2\tau_{\phi\theta}\, r\cos\phi}{r(a + r\sin\phi)} = 0 \qquad (G8)$$

C. HOOKE'S LAW INCLUDING TEMPERATURE TERMS

$$\tau_{ii} = \lambda\theta + 2\mu e_{ii} - (3\lambda + 2\mu)\int_{T_o}^{T} \alpha(T)dT \qquad (G9)$$

$$\tau_{ij} = 2\mu e_{ij} \qquad (G10)$$

where

$$\theta = e_{11} + e_{22} + e_{33}$$

and λ and μ are the Lamé constants defined in terms of Poisson's ratio ν and Young's modulus E according to

$$\left.\begin{aligned} \lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)} \\ \mu &= \frac{E}{2(1+\nu)} \end{aligned}\right\} \qquad (G11)$$

D. STRAIN - DISPLACEMENT RELATIONS

$$e_{ii} = \frac{\partial}{\partial \alpha_i}\frac{u_i}{\sqrt{g_{ii}}} + \frac{1}{2g_{ii}}\sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_k}\frac{u_k}{\sqrt{g_{kk}}} \qquad (G12)$$

$$e_{ij} = \frac{1}{2\sqrt{g_{ii}g_{jj}}}\left[g_{ii}\frac{\partial}{\partial \alpha_j}\left(\frac{u_i}{\sqrt{g_{ii}}}\right) + g_{jj}\frac{\partial}{\partial \alpha_i}\left(\frac{u_j}{\sqrt{g_{jj}}}\right)\right], \quad i \neq j \qquad (G13)$$

Let u, v, and w be components of displacement in the three principal directions r or R, ϕ and θ. Then substitution of these displacements in Equations (G12) and (G13), with the metric coefficients of Page G2, yields the strain-displacement relations for the coordinate systems shown in Sections II,E and II,F.

E. SPHERICAL COORDINATES

$$\left.\begin{aligned}
e_{RR} &= \frac{\partial u}{\partial R} \\
e_{\phi\phi} &= \frac{1}{R}\,\frac{\partial v}{\partial \phi} + \frac{u}{R} \\
e_{\theta\theta} &= \frac{1}{R\sin\phi}\,\frac{\partial w}{\partial \theta} + \frac{u}{R} + \frac{v\cot\phi}{R} \\
e_{R\phi} &= \frac{1}{2}\left(\frac{1}{R}\,\frac{\partial u}{\partial \phi} - \frac{v}{R} + \frac{\partial v}{\partial R}\right) \\
e_{\phi\theta} &= \frac{1}{2}\left(\frac{1}{R}\,\frac{\partial w}{\partial \phi} - \frac{w\cot\phi}{R} + \frac{1}{R\sin\phi}\,\frac{\partial v}{\partial \theta}\right) \\
e_{R\theta} &= \frac{1}{2}\left(\frac{1}{R\sin\phi}\,\frac{\partial u}{\partial \theta} - \frac{w}{R} + \frac{\partial w}{\partial R}\right)
\end{aligned}\right\} \qquad \text{(G14)}$$

F. TOROIDAL COORDINATES

$$\begin{aligned}
e_{rr} &= \frac{\partial u}{\partial r} \\
e_{\phi\phi} &= \frac{1}{r}\,\frac{\partial v}{\partial \phi} + \frac{u}{r} \\
e_{\theta\theta} &= \frac{1}{a + r\sin\phi}\,\frac{\partial w}{\partial \theta} + \frac{u\sin\phi}{a + r\sin\phi} + \frac{v\cos\phi}{a + r\sin\phi} \\
e_{r\phi} &= \frac{1}{2}\left(\frac{1}{r}\,\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r}\right)
\end{aligned} \qquad \text{(G15)}$$

$$e_{\phi\theta} = \frac{1}{2}\left(\frac{1}{a + r\sin\phi}\frac{\partial v}{\partial\theta} + \frac{1}{r}\frac{\partial w}{\partial\phi} - \frac{w\cos\phi}{a + r\sin\phi}\right)$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{a + r\sin\phi}\frac{\partial u}{\partial\theta} + \frac{\partial w}{\partial r} - \frac{w\sin\phi}{a + r\sin\phi}\right) \tag{G15}$$

G. EQUILIBRIUM EQUATIONS IN TERMS OF DISPLACEMENTS

Expressing the stresses in terms of displacement using Hooke's law [Equations (G9) and (G10)] with the strain-displacement relations [Equations (G14) and (G15)], the equilibrium equations [Equations (G3) through (G8)] may be written in terms of displacements in the form

$$\begin{aligned}
&A_k \frac{\partial^2 u}{\partial\alpha_1^2} + B_k \frac{\partial^2 u}{\partial\alpha_2^2} + C_k \frac{\partial^2 u}{\partial\alpha_3^2} + D_k \frac{\partial^2 u}{\partial\alpha_1\partial\alpha_2} + E_k \frac{\partial^2 u}{\partial\alpha_2\partial\alpha_3} \\
&+ F_k \frac{\partial^2 u}{\partial\alpha_1\partial\alpha_3} + G_k \frac{\partial u}{\partial\alpha_1} + H_k \frac{\partial u}{\partial\alpha_2} + I_k \frac{\partial u}{\partial\alpha_3} + J_k u \\
&+ \bar{A}_k \frac{\partial^2 v}{\partial\alpha_1^2} + \bar{B}_k \frac{\partial^2 v}{\partial\alpha_2^2} + \bar{C}_k \frac{\partial^2 v}{\partial\alpha_3^2} + \bar{D}_k \frac{\partial^2 v}{\partial\alpha_1\partial\alpha_2} + \bar{E}_k \frac{\partial^2 v}{\partial\alpha_2\partial\alpha_3} \\
&+ \bar{F}_k \frac{\partial^2 v}{\partial\alpha_1\partial\alpha_3} + \bar{G}_k \frac{\partial v}{\partial\alpha_1} + \bar{H}_k \frac{\partial v}{\partial\alpha_2} + \bar{I}_k \frac{\partial v}{\partial\alpha_3} + \bar{J}_k v \\
&+ \bar{\bar{A}}_k \frac{\partial^2 w}{\partial\alpha_1^2} + \bar{\bar{B}}_k \frac{\partial^2 w}{\partial\alpha_2^2} + \bar{\bar{C}}_k \frac{\partial^2 w}{\partial\alpha_3^2} + \bar{\bar{D}}_k \frac{\partial^2 w}{\partial\alpha_1\partial\alpha_2} + \bar{\bar{E}}_k \frac{\partial^2 w}{\partial\alpha_2\partial\alpha_3} \\
&+ \bar{\bar{F}}_k \frac{\partial^2 w}{\partial\alpha_1\partial\alpha_3} + \bar{\bar{G}}_k \frac{\partial w}{\partial\alpha_1} + \bar{\bar{H}}_k \frac{\partial w}{\partial\alpha_2} + \bar{\bar{I}}_k \frac{\partial w}{\partial\alpha_3} + \bar{\bar{J}}_k w \\
&= \frac{(3\lambda + 2\mu)\,\alpha(T)}{\sqrt{g_{kk}}}\frac{\partial T}{\partial\alpha_k}\,, \quad k = 1,2,3
\end{aligned} \tag{G16}$$

TABLE G1

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

SPHERICAL COORDINATES

$(\phi \neq 0)$

	k = 1	k = 2	k = 3
A_k	$\lambda + 2\mu$	0	0
B_k	μ/R^2	0	0
C_k	$\mu/(R^2 \sin^2 \phi)$	0	0
D_k	0	$(\lambda + \mu)/R$	0
E_k	0	0	0
F_k	0	0	$(\lambda + \mu)/(R \sin \phi)$
G_k	$2(\lambda + 2\mu)/R$	0	0
H_k	$\mu \cot \phi/R^2$	$2(\lambda + 2\mu)/R^2$	0
I_k	0	0	$2(\lambda + 2\mu)/(R^2 \sin \phi)$
J_k	$-2(\lambda + 2\mu)/R^2$	0	0
$\bar{A}_k$	0	μ	0
$\bar{B}_k$	0	$(\lambda + 2\mu)/R^2$	0
$\bar{C}_k$	0	$\mu/(R^2 \sin^2 \phi)$	0
$\bar{D}_k$	$(\lambda + \mu)/R$	0	0
$\bar{E}_k$	0	0	$(\lambda + \mu)/(R^2 \sin \phi)$
$\bar{F}_k$	0	0	0
$\bar{G}_k$	$(\lambda + \mu) \cot \phi/R$	$2\mu/R$	0
$\bar{H}_k$	$-(\lambda + 3\mu)/R^2$	$(\lambda + 2\mu) \cot \phi/R^2$	0
$\bar{I}_k$	0	0	$(\lambda + 3\mu) \cot \phi/(R^2 \sin \phi)$
$\bar{J}_k$	$-(\lambda + 3\mu) \cot \phi/R^2$	$-(\lambda + 2\mu)/(R^2 \sin^2 \phi)$	0
$\bar{\bar{A}}_k$	0	0	μ
$\bar{\bar{B}}_k$	0	0	μ/R^2

TABLE G1 (cont.)

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

SPHERICAL COORDINATES

$(\phi \neq 0)$

	k = 1	k = 2	k = 3
$\bar{\bar{C}}_k$	0	0	$(\lambda + 2\mu)/(R^2 \sin^2 \phi)$
$\bar{\bar{D}}_k$	0	0	0
$\bar{\bar{E}}_k$	0	$(\lambda + \mu)/(R^2 \sin \phi)$	0
$\bar{\bar{F}}_k$	$(\lambda + \mu)/(R \sin \phi)$	0	0
$\bar{\bar{G}}_k$	0	0	$2\mu/R$
$\bar{\bar{H}}_k$	0	0	$\mu \cot \phi/R^2$
$\bar{\bar{I}}_k$	$-(\lambda + 3\mu)/(R^2 \sin \phi)$	$-(\lambda + 3\mu) \cot \phi/(R^2 \sin \phi)$	0
$\bar{\bar{J}}_k$	0	0	$-\mu/(R^2 \sin^2 \phi)$

TABLE G2

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

TOROIDAL COORDINATES

	k = 1	k = 2	k = 3
A_k	$\lambda+2\mu$	0	0
B_k	μ/r^2	0	0
C_k	$\mu/(a+r\sin\phi)^2$	0	0
D_k	0	$(\lambda+\mu)/r$	0
E_k	0	0	0
F_k	0	0	$(\lambda+\mu)/(a+r\sin\phi)$
G_k	$(\lambda+2\mu)(a+2r\sin\phi)/\left[r(a+r\sin\phi)\right]$	0	0
H_k	$\mu\cos\phi/\left[r(a+r\sin\phi)\right]$	$\dfrac{2(\lambda+2\mu)r\sin\phi+(\lambda+3\mu)a}{r^2(a+r\sin\phi)}$	0

TABLE G2 (cont.)

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

TOROIDAL COORDINATES

	k = 1	k = 2	k = 3
I_k	0	0	$\dfrac{2(\lambda+2\mu)r\sin\phi+(\lambda+\mu)a}{r(a+r\sin\phi)^2}$
J_k	$-(\lambda+2\mu)\left[1/r^2+\sin^2\phi/(a+r\sin\phi)^2\right]$	$(\lambda+2\mu)a\cos\phi/\left[r(a+r\sin\phi)^2\right]$	0
$\bar{A}_k$	0	μ	0
$\bar{B}_k$	0	$(\lambda+2\mu)/r^2$	0
$\bar{C}_k$	0	$\mu/(a+r\sin\phi)^2$	0
$\bar{D}_k$	$(\lambda+\mu)/r$	0	0
$\bar{E}_k$	0	0	$(\lambda+\mu)/\left[r(a+r\sin\phi)\right]$
$\bar{F}_k$	0	0	0
$\bar{G}_k$	$(\lambda+\mu)\cos\phi/(a+r\sin\phi)$	$\mu(a+2r\sin\phi)/\left[r(a+r\sin\phi)\right]$	0
$\bar{H}_k$	$-(\lambda+3\mu)/r^2$	$(\lambda+2\mu)\cos\phi/\left[r(a+r\sin\phi)\right]$	0
$\bar{I}_k$	0	0	$(\lambda+3\mu)\cos\phi/(a+r\sin\phi)^2$
$\bar{J}_k$	$-\dfrac{(\lambda+3\mu)r\sin\phi\cos\phi+\mu a\cos\phi}{r(a+r\sin\phi)^2}$	$-\dfrac{(\lambda+2\mu)r^2+\mu a^2+(\lambda+3\mu)ar\sin\phi}{r^2(a+r\sin\phi)^2}$	0
$\bar{\bar{A}}_k$	0	0	μ
$\bar{\bar{B}}_k$	0	0	μ/r^2
$\bar{\bar{C}}_k$	0	0	$(\lambda+2\mu)/(a+r\sin\phi)^2$
$\bar{\bar{D}}_k$	0	0	0
$\bar{\bar{E}}_k$	0	$(\lambda+\mu)/\left[r(a+r\sin\phi)\right]$	0
$\bar{\bar{F}}_k$	$(\lambda+\mu)/(a+r\sin\phi)$	0	0
$\bar{\bar{G}}_k$	0	0	$\mu(a+2r\sin\phi)/\left[r(a+r\sin\phi)\right]$
$\bar{\bar{H}}_k$	0	0	$\mu\cos\phi/\left[r(a+r\sin\phi)\right]$
$\bar{\bar{I}}_k$	$-(\lambda+3\mu)\sin\phi/(a+r\sin\phi)^2$	$-(\lambda+3\mu)\cos\phi/(a+r\sin\phi)^2$	0
$\bar{\bar{J}}_k$	0	0	$-\mu/(a+r\sin\phi)^2$

TABLE G3

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

POLAR COORDINATES [FOR APPROXIMATE TWO-DIMENSIONAL LOCAL SOLUTION FOR ANY (θ = constant) CROSS-SECTIONAL PLANE]

	k = 1	k = 2
A_k	$\lambda + 2\mu$	0
B_k	μ/R^2	0
D_k	0	$(\lambda + \mu)/R$
G_k	$(\lambda + 2\mu)/R + \frac{\partial}{\partial T}(\lambda + 2\mu)\frac{\partial T}{\partial R}$	$\frac{1}{R}\left(\frac{\partial\lambda}{\partial T}\right)\left(\frac{\partial T}{\partial \phi}\right)$
H_k	$\frac{1}{R^2}\left(\frac{\partial\mu}{\partial T}\right)\left(\frac{\partial T}{\partial \phi}\right)$	$\frac{1}{R}\left(\frac{\partial\mu}{\partial T}\right)\left(\frac{\partial T}{\partial R}\right) + (\lambda + 3\mu)/R^2$
J_k	$\frac{1}{R}\left(\frac{\partial\lambda}{\partial T}\right)\left(\frac{\partial T}{\partial R}\right) - (\lambda + 2\mu)/R^2$	$\left(\frac{1}{R^2}\right)\left(\frac{\partial}{\partial T}\right)(\lambda + 2\mu)\frac{\partial T}{\partial \phi}$
$\bar{A}_k$	0	μ
$\bar{B}_k$	0	$(\lambda + 2\mu)/R^2$
$\bar{D}_k$	$(\lambda + \mu)/R$	0
$\bar{G}_k$	$\frac{1}{R}\left(\frac{\partial\mu}{\partial T}\right)\left(\frac{\partial T}{\partial \phi}\right)$	$\left(\frac{\partial\mu}{\partial T}\right)\left(\frac{\partial T}{\partial R}\right) + \mu/R$
$\bar{H}_k$	$\frac{1}{R}\left(\frac{\partial\lambda}{\partial T}\right)\left(\frac{\partial T}{\partial R}\right) - \frac{(\lambda + 3\mu)}{R^2}$	$\left(\frac{1}{R^2}\right)\left(\frac{\partial}{\partial T}\right)(\lambda + 2\mu)\frac{\partial T}{\partial \phi}$
$\bar{J}_k$	$-\frac{1}{R^2}\left(\frac{\partial\mu}{\partial T}\right)\left(\frac{\partial T}{\partial \phi}\right)$	$-\frac{1}{R}\left(\frac{\partial\mu}{\partial T}\right)\left(\frac{\partial T}{\partial R}\right) - \mu/R^2$

Note: Temperature-dependent material property derivative terms are also included. Only applicable coefficients are listed. By replacing R with r, the above coefficients are applicable in the torus cross-section region.

H. EQUATIONS FOR STRESSES IN TERMS OF DISPLACEMENTS

From Hooke's law, Equations (G9) and (G10)

$$\tau_{ij} = 2\mu e_{ij} + \delta_{ij} \left[\lambda\theta - (3\lambda + 2\mu) \int_{T_o}^{T} \alpha(T)dT\right] \tag{G17}$$

where δ_{ij} is the Kronecker delta defined by

$$\begin{aligned} \delta_{ij} &= 1, \; i = j \\ &= 0, \; i \neq j \end{aligned}$$

and

$$\theta \equiv e_{11} + e_{22} + e_{33}$$

Writing the strains in terms of displacements from either Equation (G14) or (G15) and shortening the nomenclature by defining the stresses

$$\tau_1 = \tau_{rr} \text{ or } \tau_{RR}$$

$$\tau_2 = \tau_{\phi\phi}$$

$$\tau_3 = \tau_{\theta\theta}$$

$$\tau_4 = \tau_{r\phi} \text{ or } \tau_{R\phi}$$

$$\tau_5 = \tau_{\phi\theta}$$

$$\tau_6 = \tau_{r\theta} \text{ or } \tau_{R\theta}$$

Equation (G17) may be written in terms of displacements according to

$$\begin{aligned} \tau_\ell + \Delta_\ell (3\lambda + 2\mu) \int_{T_o}^{T} \alpha(T)dT = {} & \alpha_\ell u_r + \beta_\ell u_\phi + \gamma_\ell u_\theta + \delta_\ell u \\ & + \bar{\alpha}_\ell v_r + \bar{\beta}_\ell v_\phi + \bar{\gamma}_\ell v_\theta + \bar{\delta}_\ell v \\ & + \bar{\bar{\alpha}}_\ell w_r + \bar{\bar{\beta}}_\ell w_\phi + \bar{\bar{\gamma}}_\ell w_\theta + \bar{\bar{\delta}}_\ell w \end{aligned} \tag{G18}$$

where

$$\Delta_\ell = 1 \quad \text{if} \quad \ell = 1,2,3$$
$$= 0 \quad \text{if} \quad \ell = 4,5,6$$

TABLE G4

COEFFICIENTS OF STRESS EQUATIONS

SPHERICAL COORDINATES

($\phi \neq 0$)

ℓ	1	2	3	4	5	6
α_ℓ	$\lambda + 2\mu$	λ	λ	0	0	0
β_ℓ	0	0	0	μ/R	0	0
γ_ℓ	0	0	0	0	0	$\mu/(R \sin \phi)$
δ_ℓ	$2\lambda/R$	$2(\lambda + \mu)/R$	$2(\lambda + \mu)/R$	0	0	0
$\bar{\alpha}_\ell$	0	0	0	μ	0	0
$\bar{\beta}_\ell$	λ/R	$(\lambda + 2\mu)/R$	λ/R	0	0	0
$\bar{\gamma}_\ell$	0	0	0	0	$\mu/(R \sin \phi)$	0
$\bar{\delta}_\ell$	$\lambda \cot \phi/R$	$\lambda \cot \phi/R$	$(\lambda + 2\mu) \cot \phi/R$	$-\mu/R$	0	0
$\bar{\bar{\alpha}}_\ell$	0	0	0	0	0	μ
$\bar{\bar{\beta}}_\ell$	0	0	0	0	μ/R	0
$\bar{\bar{\gamma}}_\ell$	$\lambda/(R \sin \phi)$	$\lambda/(R \sin \phi)$	$(\lambda + 2\mu)/(R \sin \phi)$	0	0	0
$\bar{\bar{\delta}}_\ell$	0	0	0	0	$-\mu \cot \phi/R$	$-\mu/R$

TABLE G5

COEFFICIENTS OF STRESS EQUATIONS

TOROIDAL COORDINATES

ℓ	1	2	3	4	5	6
α_ℓ	$\lambda+2\mu$	λ	λ	0	0	0
β_ℓ	0	0	0	μ/R	0	0
γ_ℓ	0	0	0	0	0	$\mu/(a+r\sin\phi)$
δ_ℓ	$\frac{\lambda(a+2r\sin\phi)}{r(a+r\sin\phi)}$	$\frac{\lambda+2\mu}{r}+\frac{\lambda\sin\phi}{a+r\sin\phi}$	$\frac{\lambda}{r}+\frac{(\lambda+2\mu)\sin\phi}{a+r\sin\phi}$	0	0	0
$\bar{\alpha}_\ell$	0	0	0	μ	0	0
$\bar{\beta}_\ell$	λ/r	$(\lambda+2\mu)/r$	λ/r	0	0	0
$\bar{\gamma}_\ell$	0	0	0	0	$\mu/(a+r\sin\phi)$	0
$\bar{\delta}_\ell$	$\frac{\lambda\cos\phi}{a+r\sin\phi}$	$\frac{\lambda\cos\phi}{a+r\sin\phi}$	$\frac{(\lambda+2\mu)\cos\phi}{a+r\sin\phi}$	$-\mu/r$	0	0
$\bar{\bar{\alpha}}_\ell$	0	0	0	0	0	μ
$\bar{\bar{\beta}}_\ell$	0	0	0	0	μ/r	0
$\bar{\bar{\gamma}}_\ell$	$\lambda/(a+r\sin\phi)$	$\frac{\lambda}{a+r\sin\phi}$	$\frac{\lambda+2\mu}{a+r\sin\phi}$	0	0	0
$\bar{\bar{\delta}}_\ell$	0	0	0	0	$-\frac{\mu\cos\phi}{a+r\sin\phi}$	$-\frac{\mu\sin\phi}{a+r\sin\phi}$

TABLE G6

COEFFICIENTS OF EQUILIBRIUM EQUATIONS

POLAR COORDINATES [FOR APPROXIMATE TWO-DIMENSIONAL LOCAL SOLUTION FOR ANY (θ = constant) CROSS-SECTIONAL PLANE]

	1	2	4
α_ℓ	$\lambda + 2\mu$	λ	0
β_ℓ	0	0	μ/R
δ_ℓ	λ/R	$(\lambda + 2\mu)/R$	0
$\bar{\alpha}_\ell$	0	0	μ
$\bar{\beta}_\ell$	λ/R	$(\lambda + 2\mu)/R$	0
$\bar{\delta}_\ell$	0	0	$-\mu/R$

Note: Only applicable values of ℓ and coefficients are listed. By replacing R with r, the above coefficients are applicable in the torus cross-section region.

I. EQUATIONS AT THE AXIS OF SYMMETRY

Certain of the coefficients in the displacement equilibrium and stress equations become singular at the axis of symmetry ($\phi = 0$). For the nonaxially symmetric case the axis of symmetry has no special physical significance and this point can be avoided. For the axially symmetric case, however, the axis of symmetry is generally quite important and the singular coefficients may be evaluated by the use of L'Hôspital's rule. For example, the coefficient H_1 in the displacement equilibrium equations in spherical coordinates is $\mu \cot \phi / R^2$ which becomes infinite as ϕ approaches zero. From Equation (G16), this term multiplies the displacement component $\frac{\partial u}{\partial \phi}$. The conditions for axial symmetry are

$$w(R,\phi,\theta) = \frac{\partial f}{\partial \theta} = 0 \qquad \text{(G19)}$$

where f is any function of R,ϕ,θ. From this it can be shown that

$$v = \frac{\partial u}{\partial \phi} = \frac{\partial^2 v}{\partial \phi^2} = 0 \text{ at } \phi = 0. \qquad \text{(G20)}$$

Hence, since $\frac{\partial u}{\partial \phi}$ approaches zero while H_1 approaches infinity, L'Hôspital's rule is applicable to the product

$$\frac{\mu \cot \phi}{R^2} \frac{\partial u}{\partial \phi}$$

as $\phi \longrightarrow 0$. Taking the limit, there is obtained

$$\lim_{\phi \to 0} \frac{\mu \cot \phi}{R^2} \cdot \frac{\partial u}{\partial \phi} = \frac{\mu}{R^2} \lim_{\phi \to 0} \frac{\frac{\partial u}{\partial \phi} \cos \phi}{\sin \phi}$$

$$= \frac{\mu}{R^2} \lim_{\phi \to 0} \frac{\frac{\partial^2 u}{\partial \phi^2} \cos \phi - \frac{\partial u}{\partial \phi} \sin \phi}{\cos \phi}$$

$$= \frac{\mu}{R^2} \cdot \frac{\partial^2 u}{\partial \phi^2}$$

Hence, for this case, the coefficient H_1 becomes zero and the coefficient B_1 which multiplies $\frac{\partial^2 u}{\partial \phi^2}$ is increased by μ/R^2. Applying this limiting process to all the singular terms, the following sets of coefficients are obtained:

TABLE G7

COEFFICIENTS OF EQUILIBRIUM EQUATIONS ON AXIS OF SYMMETRY (ϕ = 0) FOR AXIALLY SYMMETRIC CASE SPHERICAL COORDINATES

k	1	2	3	k	1	2	3	k	1	2	3
A_k	$\lambda+2\mu$	0	0	$\bar{A}_k$	0	0	0	$\bar{\bar{A}}_k$	0	0	0
B_k	$2\mu/R^2$	0	0	$\bar{B}_k$	0	0	0	$\bar{\bar{B}}_k$	0	0	0
C_k	0	0	0	$\bar{C}_k$	0	0	0	$\bar{\bar{C}}_k$	0	0	0
D_k	0	0	0	$\bar{D}_k$	$2(\lambda+\mu)/R$	0	0	$\bar{\bar{D}}_k$	0	0	0
E_k	0	0	0	$\bar{E}_k$	0	0	0	$\bar{\bar{E}}_k$	0	0	0
F_k	0	0	0	$\bar{F}_k$	0	0	0	$\bar{\bar{F}}_k$	0	0	0
G_k	$2(\lambda+2\mu)/R$	0	0	$\bar{G}_k$	0	0	0	$\bar{\bar{G}}_k$	0	0	0
H_k	0	0	0	$\bar{H}_k$	$-2(\lambda+3\mu)/R^2$	0	0	$\bar{\bar{H}}_k$	0	0	0
I_k	0	0	0	$\bar{I}_k$	0	0	0	$\bar{\bar{I}}_k$	0	0	0
J_k	$-2(\lambda+2\mu)/R^2$	0	0	$\bar{J}_k$	0	0	0	$\bar{\bar{J}}_k$	0	0	0

TABLE G8

COEFFICIENTS OF STRESS EQUATIONS ON AXIS OF SYMMETRY (ϕ = 0) FOR AXIALLY SYMMETRIC CASE SPHERICAL COORDINATES

ℓ	1	2	3	4	5	6
α_ℓ	$\lambda + 2\mu$	λ	λ	0	0	0
β_ℓ	0	0	0	μ/R	0	0
γ_ℓ	0	0	0	0	0	0
δ_ℓ	$2\lambda/R$	$2(\lambda + \mu)/R$	$2(\lambda + \mu)/R$	0	0	0
$\bar{\alpha}_\ell$	0	0	0	μ	0	0
$\bar{\beta}_\ell$	$2\lambda/R$	$2(\lambda + \mu)/R$	$2(\lambda + \mu)/R$	0	0	0
$\bar{\gamma}_\ell$	0	0	0	0	0	0
$\bar{\delta}_\ell$	0	0	0	$-\mu/R$	0	0
$\bar{\bar{\alpha}}_\ell$	0	0	0	0	0	0
$\bar{\bar{\beta}}_\ell$	0	0	0	0	0	0
$\bar{\bar{\gamma}}_\ell$	0	0	0	0	0	0
$\bar{\bar{\delta}}_\ell$	0	0	0	0	0	0

J. TEMPERATURE DEPENDENCE OF ELASTIC CONSTANTS

If, in addition to the coefficient of thermal expansion, the elastic constants are strongly dependent on temperature, then additional terms must be included in the displacement equilibrium equations to account for the special derivatives of these constants. For example, in differentiating the stress component τ_{ii} with respect to coordinate α_i [from Equation (G9)] there is obtained

$$\left.\begin{aligned}
\frac{\partial \tau_{ii}}{\partial \alpha_i} &= \lambda \frac{\partial \theta}{\partial \alpha_i} + \theta \frac{\partial \lambda}{\partial \alpha_i} + 2\mu \frac{\partial e_{ii}}{\partial \alpha_i} + 2\, e_{ii} \frac{\partial \mu}{\partial \alpha_i} \\
&\quad - (3\lambda + 2\mu)\, \alpha(T) \frac{\partial T}{\partial \alpha_i} - \frac{\partial}{\partial \alpha_i} - (3\lambda + 2\mu) \int_{T_o}^{T} \alpha(T)\, dT \\
&= \theta \left|\frac{\partial \lambda}{\partial T}\right| \left|\frac{\partial T}{\partial \alpha_i}\right| + 2\, e_{ii} \left|\frac{\partial \mu}{\partial T}\right| \left|\frac{\partial T}{\partial \alpha_i}\right| - \frac{\partial}{\partial T} (3\lambda + 2\mu) \frac{\partial T}{\partial \alpha_i} \int_{T_o}^{T} \alpha(T)\, dT \\
&\quad + \lambda \frac{\partial \theta}{\partial \alpha_i} + 2\mu \frac{\partial e_{ii}}{\partial \alpha_i} - (3\lambda + 2\mu)\, \alpha(T) \frac{\partial T}{\partial \alpha_i}
\end{aligned}\right\} \qquad \text{(G21)}$$

where the first three terms to the right of the equal sign have not been accounted for in the coefficients of Equation (G16). Representing the additional terms by primed quantities, Equation (G16) becomes

$$(A_k + A_k') \frac{\partial^2 u}{\partial \alpha_1{}^2} + (B_k + B_k') \frac{\partial^2 u}{\partial \alpha_2{}^2} + \ldots = \frac{(3\lambda + 2\mu)\, \alpha(T)}{\sqrt{g_{kk}}} \frac{\partial T}{\partial \alpha_k}$$

$$+ \frac{1}{\sqrt{g_{kk}}} \frac{\partial}{\partial T} (3\lambda + 2\mu) \frac{\partial T}{\partial \alpha_k} \int_{T_o}^{T} \alpha(T)\, dT, \; k = 1, 2, 3 \qquad \text{(G22)}$$

The coefficients A_k', B_k', ... are tabulated below for spherical and toroidal coordinates, and for the special point in spherical coordinates on the axis of symmetry for the case of axial symmetry.

TABLE G9

ADDITIONAL TERMS IN COEFFICIENTS OF EQUILIBRIUM EQUATIONS FROM TEMPERATURE DEPENDENCE OF ELASTIC CONSTANTS SPHERICAL COORDINATES

	k = 1	k = 2	k = 3
A'_k	0	0	0
B'_k	0	0	0
C'_k	0	0	0
D'_k	0	0	0
E'_k	0	0	0
F'_k	0	0	0
G'_k	$\frac{\partial}{\partial T}(\lambda+2\mu)\frac{\partial T}{\partial R}$	$\frac{1}{R}\left(\frac{\partial \lambda}{\partial T}\right)\left(\frac{\partial T}{\partial \phi}\right)$	$\frac{1}{R\sin\phi}\left(\frac{\partial \lambda}{\partial T}\right)\frac{\partial T}{\partial \theta}$
H'_k	$\frac{1}{R^2}\left(\frac{\partial \mu}{\partial T}\right)\left(\frac{\partial T}{\partial \phi}\right)$	$\frac{1}{R}\left(\frac{\partial \mu}{\partial T}\right)\left(\frac{\partial T}{\partial R}\right)$	0
I'_k	$\frac{1}{R^2\sin^2\phi}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial \theta}$	0	$\frac{1}{R\sin\phi}\left(\frac{\partial \mu}{\partial T}\right)\frac{\partial T}{\partial R}$
J'_k	$\frac{2}{R}\left(\frac{\partial \lambda}{\partial T}\right)\left(\frac{\partial T}{\partial R}\right)$	$\frac{2}{R^2}\left(\frac{\partial}{\partial T}\right)(\lambda+\mu)\frac{\partial T}{\partial \phi}$	$\frac{2}{R^2\sin\phi}\left(\frac{\partial}{\partial T}\right)(\lambda+\mu)\frac{\partial T}{\partial \theta}$
$\bar{A}'_k$	0	0	0
$\bar{B}'_k$	0	0	0
$\bar{C}'_k$	0	0	0
$\bar{D}'_k$	0	0	0
$\bar{E}'_k$	0	0	0
$\bar{F}'_k$	0	0	0
$\bar{G}'_k$	$\frac{1}{R}\left(\frac{\partial \mu}{\partial T}\right)\left(\frac{\partial T}{\partial \phi}\right)$	$\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial R}$	0

TABLE G9 (cont.)

ADDITIONAL TERMS IN COEFFICIENTS OF EQUILIBRIUM EQUATIONS
FROM TEMPERATURE DEPENDENCE OF ELASTIC CONSTANTS
SPHERICAL COORDINATES

	k = 1	k = 2	k = 3
$\bar{H}'_k$	$\frac{1}{R}\left(\frac{\partial\lambda}{\partial T}\right)\left(\frac{\partial T}{\partial R}\right)$	$\frac{1}{R^2}\left(\frac{\partial}{\partial T}\right)(\lambda+2\mu)\frac{\partial T}{\partial\phi}$	$\frac{1}{R^2\sin\phi}\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial\theta}$
$\bar{I}'_k$	0	$\frac{1}{R^2\sin^2\phi}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial\theta}$	$\frac{1}{R^2\sin\phi}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial\phi}$
$\bar{J}'_k$	$\frac{\partial\lambda}{\partial T}\left(\frac{\partial T}{\partial R}\right)\left(\frac{\cot\phi}{R}\right) - \frac{1}{R^2}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial\phi}$	$\frac{\cot\phi}{R^2}\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial\phi} - \frac{1}{R}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial R}$	$\frac{\cot\phi}{R^2\sin\phi}\left(\frac{\partial}{\partial T}\right)(\lambda+2\mu)\frac{\partial T}{\partial\theta}$
$\bar{\bar{A}}'_k$	0	0	0
$\bar{\bar{B}}'_k$	0	0	0
$\bar{\bar{C}}'_k$	0	0	0
$\bar{\bar{D}}'_k$	0	0	0
$\bar{\bar{E}}'_k$	0	0	0
$\bar{\bar{F}}'_k$	0	0	0
$\bar{\bar{G}}'_k$	$\frac{1}{R\sin\phi}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial\theta}$	0	$\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial R}$
$\bar{\bar{H}}'_k$	0	$\frac{1}{R^2\sin\phi}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial\theta}$	$\frac{1}{R^2}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial\phi}$
$\bar{\bar{I}}'_k$	$\frac{1}{R\sin\phi}\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial R}$	$\frac{1}{R^2\sin\phi}\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial\phi}$	$\frac{1}{R^2\sin^2\phi}\left(\frac{\partial}{\partial T}\right)(\lambda+2\mu)\frac{\partial T}{\partial\theta}$
$\bar{\bar{J}}'_k$	$-\frac{1}{R^2\sin\phi}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial\theta}$	$-\frac{\cot\phi}{R^2\sin\phi}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial\theta}$	$-\frac{1}{R}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial R} - \frac{\cot\phi}{R^2}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial\phi}$

TABLE G10

ADDITIONAL TERMS IN COEFFICIENTS OF EQUILIBRIUM EQUATIONS FROM TEMPERATURE DEPENDENCE OF ELASTIC CONSTANTS TOROIDAL COORDINATES

	k = 1	k = 2	k = 3
A_k'	0	0	0
B_k'	0	0	0
C_k'	0	0	0
D_k'	0	0	0
E_k'	0	0	0
F_k'	0	0	0
G_k'	$\frac{\partial}{\partial T}(\lambda+2\mu)\frac{\partial T}{\partial r}$	$\frac{1}{r}\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial \phi}$	$\frac{1}{(a+r\sin\phi)}\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial \theta}$
H_k'	$\frac{1}{r^2}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial \phi}$	$\frac{1}{r}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial r}$	0
I_k'	$\frac{1}{(a+r\sin\phi)^2}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial \theta}$	0	$\frac{1}{(a+r\sin\phi)}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial r}$
J_k'	$\left[\frac{1}{r}+\frac{\sin\phi}{(a+r\sin\phi)}\right]\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial r}$	$\frac{1}{r^2}\left(\frac{\partial}{\partial T}\right)(\lambda+2\mu)\frac{\partial T}{\partial \phi} + \frac{\sin\phi}{r(a+r\sin\phi)}\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial \phi}$	$\frac{\sin\phi}{(a+r\sin\phi)^2}\left(\frac{\partial}{\partial T}\right)(\lambda+2\mu)\frac{\partial T}{\partial \theta} + \frac{\frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial \theta}}{r(a+r\sin\phi)}$
$\bar{A}_k'$	0	0	0
$\bar{B}_k'$	0	0	0
$\bar{C}_k'$	0	0	0
$\bar{D}_k'$	0	0	0
$\bar{E}_k'$	0	0	0
$\bar{F}_k'$	0	0	0
$\bar{G}_k'$	$\frac{1}{r}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial \phi}$	$\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial r}$	0

<u>TABLE G10 (cont.)</u>

ADDITIONAL TERMS IN COEFFICIENTS OF EQUILIBRIUM EQUATIONS
FROM TEMPERATURE DEPENDENCE OF ELASTIC CONSTANTS
TOROIDAL COORDINATES

	k = 1	k = 2	k = 3
$\bar{H}'_k$	$\frac{1}{r}\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial r}$	$\frac{1}{r^2}\left(\frac{\partial}{\partial T}\right)(\lambda+2\mu)\frac{\partial T}{\partial \phi}$	$\frac{1}{r(a+\sin\phi)}\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial \theta}$
$\bar{I}'_k$	0	$\frac{1}{(a+r\sin\phi)^2}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial \theta}$	$\frac{1}{r(a+r\sin\phi)}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial \phi}$
$\bar{J}'_k$	$\frac{\cos\phi}{(a+r\sin\phi)}\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial r} - \frac{1}{r^2}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial \phi}$	$\frac{\cos\phi}{r(a+r\sin\phi)}\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial \phi} - \frac{1}{r}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial r}$	$\frac{\cos\phi}{(a+r\sin\phi)^2}\left(\frac{\partial}{\partial T}\right)(\lambda+2\mu)\frac{\partial T}{\partial \theta}$
$\bar{\bar{A}}'_k$	0	0	0
$\bar{\bar{B}}'_k$	0	0	0
$\bar{\bar{C}}'_k$	0	0	0
$\bar{\bar{D}}'_k$	0	0	0
$\bar{\bar{E}}'_k$	0	0	0
$\bar{\bar{F}}'_k$	0	0	0
$\bar{\bar{G}}'_k$	$\frac{1}{(a+r\sin\phi)}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial \theta}$	0	$\frac{\partial\mu}{\partial T}\,\frac{\partial T}{\partial r}$
$\bar{\bar{H}}'_k$	0	$\frac{1}{r(a+r\sin\phi)}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial \theta}$	$\frac{1}{r^2}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial \phi}$
$\bar{\bar{I}}'_k$	$\frac{1}{(a+r\sin\phi)}\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial r}$	$\frac{1}{r(a+r\sin\phi)}\left(\frac{\partial\lambda}{\partial T}\right)\frac{\partial T}{\partial \phi}$	$\frac{1}{(a+r\sin\phi)^2}\left(\frac{\partial}{\partial T}\right)(\lambda+2\mu)\frac{\partial T}{\partial \theta}$
$\bar{\bar{J}}'_k$	$-\frac{\sin\phi}{(a+r\sin\phi)^2}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial \theta}$	$-\frac{\cos\phi}{(a+r\sin\phi)^2}\left(\frac{\partial\mu}{\partial T}\right)\frac{\partial T}{\partial \theta}$	$-\frac{\frac{\partial\mu}{\partial T}}{(a+r\sin\phi)}\left[\frac{\partial T}{\partial r}(\sin\phi) + \frac{\partial T}{\partial \phi}\left(\frac{\cos\phi}{r}\right)\right]$

K. AXIS OF SYMMETRY WITH AXIAL SYMMETRY

The only non-zero terms in the coefficients of Table G9 on the axis of symmetry in the axially symmetric case are the following:

$$G_1' = \frac{\partial}{\partial T} (\lambda + 2\mu) \frac{\partial T}{\partial R}$$

$$J_1' = \frac{2}{R} \left(\frac{\partial \lambda}{\partial T}\right) \frac{\partial T}{\partial R}$$

$$\overline{H}_1' = \frac{2}{R} \left(\frac{\partial \lambda}{\partial R}\right) \frac{\partial T}{\partial R}$$

The integral term in Equation (G22) is also non-zero for the equilibrium equation corresponding to k = 1.

L. EQUATIONS FOR THE SINGULAR POINT, $\phi = 0$, FOR THE NON-AXISYMMETRIC CASE

The components of displacement u, v, w in the R, ϕ, θ directions, respectively, have the following properties at $\phi = 0$:

$$\left.\begin{aligned}
&\frac{\partial u}{\partial \theta} = 0 &\qquad& \frac{\partial^3 u}{\partial \theta^2 \partial \phi} = - \frac{\partial u}{\partial \phi} \\
&\frac{\partial w}{\partial \theta} = v && \frac{\partial^3 w}{\partial \theta^2 \partial \phi} = - 2 \frac{\partial^2 v}{\partial \theta \partial \phi} \\
&\frac{\partial v}{\partial \theta} = w && \frac{\partial^3 v}{\partial \theta^2 \partial \phi} = 2 \frac{\partial^2 w}{\partial \theta \partial \phi} \\
&\frac{\partial \beta}{\partial \theta} = 0 && \beta = \int_{T_o}^{T} \alpha(T)\, \partial T
\end{aligned}\right\} \qquad \text{(G23)}$$

These results are derived beginning on page G26 of this appendix.

M. EVALUATION OF STRAINS

From Equation (G23) the singular terms in the strain-displacement equations can be evaluated. Thus we find

$$e_{\theta\theta} = \frac{1}{R \sin \phi} \frac{\partial w}{\partial \theta} + \frac{u}{R} + \frac{v \cot \phi}{R} \xrightarrow[\phi \to 0]{} \frac{1}{R} \left(\frac{\frac{\partial w}{\partial \theta} + v}{\sin \phi} \right) + \frac{u}{R}$$

and from Equation (G23) we note that the bracketed part is $\left(\frac{0}{0}\right)$ and thus we have the result

$$e_{\theta\theta} = \frac{1}{R} \left(\frac{\partial^2 w}{\partial \theta \partial \phi} + \frac{\partial v}{\partial \phi} + u \right) \quad (\phi = 0) \tag{G24}$$

Similarly we find

$$e_{\phi\theta} = \frac{1}{2R} \left(\frac{\partial w}{\partial \phi} - w \cot \phi + \frac{1}{\sin \phi} \frac{\partial v}{\partial \theta} \right)$$

and again through Equation (G23) we obtain

$$\left.\begin{aligned} e_{\phi\theta} &= \frac{1}{2R} \left(\frac{\partial w}{\partial \phi} - \frac{\partial w}{\partial \phi} + \frac{\partial^2 v}{\partial \theta \partial \phi} \right) \quad (\phi = 0) \\ e_{\phi\theta} &= \frac{1}{2R} \frac{\partial^2 v}{\partial \theta \partial \phi} \quad (\phi = 0) \end{aligned}\right\} \tag{G25}$$

Finally,

$$\left.\begin{aligned} e_{R\theta} &= \frac{1}{2R} \left(\frac{1}{\sin \phi} \frac{\partial u}{\partial \theta} - w + R \frac{\partial u}{\partial R} \right) \\ e_{R\theta} &= \frac{1}{2R} \left(\frac{\partial^2 u}{\partial \theta \partial \phi} - w + R \frac{\partial u}{\partial R} \right) \quad (\phi = 0) \end{aligned}\right\} \tag{G26}$$

N. EQUATIONS OF EQUILIBRIUM

The equations of equilibrium

$$\frac{\partial \tau_{RR}}{\partial R} + \frac{1}{R} \frac{\partial \tau_{R\phi}}{\partial \phi} + \frac{1}{R \sin \phi} \frac{\partial \tau_{R\theta}}{\partial \theta} + \frac{2\tau_{RR} - \tau_{\phi\phi} - \tau_{\theta\theta} + \tau_{R\phi} \cot \phi}{R} = 0 \tag{G27}$$

$$\frac{\partial \tau_{R\phi}}{\partial R} + \frac{1}{R} \frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{1}{R \sin \phi} \frac{\partial \tau_{\phi\phi}}{\partial \theta} + \frac{3\tau_{R\phi} + (\tau_{\phi\phi} - \tau_{\theta\theta}) \cot \phi}{R} = 0 \tag{G28}$$

$$\frac{\partial \tau_{R\theta}}{\partial R} + \frac{1}{R} \frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{1}{R \sin \phi} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{3\tau_{R\theta} + 2\tau_{\phi\theta} \cot \phi}{R} = 0 \tag{G29}$$

have the indeterminate parts indicated below.

From Equation (G27), we have

$$\frac{1}{R \sin \phi} \frac{\partial \tau_{R\theta}}{\partial \theta} + \frac{\tau_{R\phi} \cot \phi}{R} \tag{G30}$$

Putting in the values of $\tau_{R\theta}$ and $\tau_{R\phi}$ in terms of displacements, we get

$$\frac{\partial \tau_{R\theta}}{\partial \theta} + \tau_{R\phi} = \frac{\mu}{R} \left[\frac{1}{\sin \phi} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial u}{\partial \phi} - \left(\frac{\partial w}{\partial \theta} + v \right) + \frac{\partial}{\partial R} \left(\frac{\partial w}{\partial \theta} + v \right) \right] \tag{G31}$$

From Equation (G23) we find that the expression (G31) is zero at $\phi = 0$. Thus we may write, for Equation (G30)

$$\frac{1}{R} \left(\frac{\partial^2 \tau_{R\theta}}{\partial \theta \partial \phi} + \frac{\partial \tau_{R\phi}}{\partial \phi} \right)$$

which is the evaluation of the indeterminate portion of Equation (G27). From Equation (G28), the indeterminate part is

$$\frac{1}{R \sin \phi} \frac{\partial \tau_{\phi\theta}}{\partial \theta} + \frac{\tau_{\theta\phi} - \tau_{\theta\theta}}{R} \cot \phi \tag{G32}$$

In terms of the displacements

$$\frac{\partial \tau_{\theta\phi}}{\partial \theta} + \tau_{\phi\phi} - \tau_{\theta\theta} = \frac{\mu}{R} \left[\frac{\partial^2 w}{\partial \theta \partial \phi} + 2 \frac{\partial v}{\partial \phi} + \frac{1}{\sin \phi} \frac{\partial}{\partial \theta} \left(\frac{\partial v}{\partial \theta} - w \right) \right.$$

$$\left. - \frac{2}{\sin \phi} \left(\frac{\partial w}{\partial \theta} + v \right) \right]$$

From Equation (G23), the terms in parenthesis are zero. Thus we have

$$\frac{\mu}{R}\left[\frac{\partial^2 w}{\partial\theta\partial\phi} + \frac{\partial^3 v}{\partial\theta^2\partial\phi} - \frac{\partial^2 w}{\partial\phi\partial\theta} - 2\,\frac{\partial^2 w}{\partial v\partial\phi}\right] = \frac{\mu}{R}\left(\frac{\partial^3 v}{\partial\theta^2\partial\phi} - 2\,\frac{\partial^2 w}{\partial\theta\partial\phi}\right)$$

which is zero by Equation (G23). Therefore we may write Equation (G32) as

$$\frac{1}{R}\;\frac{\partial^2 \tau_{\phi\theta}}{\partial\theta\partial\phi} + \frac{1}{R}\left(\frac{\partial\tau_{\phi\phi}}{\partial\phi} - \frac{\partial\tau_{\theta\theta}}{\partial\phi}\right)$$

which is the evaluation of the indeterminate part of Equation (G28).

From Equation (G29), the indeterminate parts are

$$\frac{1}{R\sin\phi}\;\frac{\partial\tau_{\theta\theta}}{\partial\theta} + 2\,\frac{\tau_{\phi\theta}}{R}\cot\phi \tag{G33}$$

and

$$\frac{1}{R}\;\frac{\partial\tau_{\theta\theta}}{\partial\theta} \tag{G34}$$

For Equation (33) we have (at $\phi \longrightarrow 0$)

$$\frac{\partial\tau_{\theta\theta}}{\partial\theta} + 2\,\tau_{\theta\phi} = \frac{2\mu}{R}\;\frac{\partial w}{\partial\phi} + \frac{2(\lambda + \mu)}{R}\;\frac{\partial u}{\partial\theta} + \lambda\,\frac{\partial^2 u}{\partial R\partial\theta} + \frac{\lambda}{R}\;\frac{\partial^2 v}{\partial\theta\partial\phi} - \frac{\partial\beta(T)}{\partial\theta}$$

$$+ \frac{2\mu}{R\sin\phi}\left(\frac{\partial v}{\partial\theta} - w\right) - \frac{(\lambda + 2\mu)}{R\sin\phi}\;\frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial\theta} + v\right)$$

It can be seen from Equation (G23) that the terms with $\frac{1}{\sin\phi}$ have the form $\frac{0}{0}$ and that the terms in u and β are zero. Thus we can write

$$\frac{\partial\tau_{\theta\theta}}{\partial\theta} + 2\,\tau_{\theta\phi} = \frac{1}{R}\left(2\mu\,\frac{\partial^2 v}{\partial\theta\partial\phi} + (\lambda + 2\mu)\,\frac{\partial^3 w}{\partial\theta^2\partial\phi} + (\lambda + 2\mu)\,\frac{\partial^2 v}{\partial\theta\partial\phi} + \lambda\,\frac{\partial^2 v}{\partial\theta\partial\phi}\right)$$

$$= \frac{\lambda + 2\mu}{R}\left(\frac{\partial^3 w}{\partial\theta^2\partial\phi} + 2\,\frac{\partial^2 v}{\partial\theta\partial\phi}\right) = 0 \quad \text{by Equation (G23)}$$

Thus we may write Equation (G32) as

$$\frac{1}{R}\left(\frac{\partial^2\tau_{\theta\theta}}{\partial\theta\partial\phi} + 2\,\frac{\partial\tau_{\theta\phi}}{\partial\phi}\right)$$

For Equation (G34) we expand the functions in powers of ϕ thus:

$$w = w_o + w_1\,\phi + w_2\,\phi^2 + \ldots$$

Then we may write (for $\phi \rightarrow 0$)

$$\frac{\partial\tau_{\theta\phi}}{\partial\phi} = \frac{\partial}{\partial\phi}\left[w_1 + 2w_2\phi - \frac{w_o + w_1\phi + w_2\phi^2}{\phi} + \left(\frac{\partial v_o}{\partial\theta} + \frac{\partial v_1}{\partial\theta}\,\phi + \frac{\partial v_2}{\partial\theta}\,\phi^2\right)\Big/\phi\right]$$

$$= 2w_2 + \frac{w_o - \frac{\partial v_o}{\partial\theta}}{\phi^2} - w_2 + \frac{\partial v_2}{\partial\theta}$$

And since $w_o = \frac{\partial v_o}{\partial\theta}$ from Equation (G23), we have

$$\left.\begin{aligned} \frac{\partial\tau_{\theta\phi}}{\partial\phi} &= w_2 + \frac{\partial v_2}{\partial\theta} \qquad (\phi = 0) \\ &= \frac{1}{2}\left(\frac{\partial^2 w}{\partial\phi^2} + \frac{\partial^3 v}{\partial\theta\partial\phi^2}\right) \qquad (\phi = 0) \end{aligned}\right\} \qquad \text{(G35)}$$

Thus, the one term, $\frac{\partial\tau_{\theta\phi}}{\partial\phi}$, in the equilibrium equations is not evaluated by simple differentiation with respect to ϕ.*

O. RELATIONS BETWEEN DISPLACEMENTS AT $\phi = 0$

To obtain the equations given in Equation (G23), we take the gradient and the Laplacian of the displacement vector, i.e.,

*See note on page G28.

$$\nabla \vec{\xi} = \nabla(\hat{R}u + \hat{\phi}v + \hat{\theta}w)$$

and

$$\nabla^2 \vec{\xi} = \nabla^2 (\hat{R}u + \hat{\phi}v + \hat{\theta}w)$$

Expanding in powers of ϕ, viz,

$$u = u_o + u_1 \phi + u_2 \phi^2 + \ldots$$

and letting $\phi = \rightarrow 0$ after obtaining $\nabla \xi$ and $\nabla^2 \xi$, we find the following terms involving $\frac{1}{\phi}$ as a factor:

$$\hat{\theta}\hat{\theta} \left(\frac{\partial w_o}{\partial \theta} + v_o \right)$$

$$\hat{\theta}\hat{\phi} \left(\frac{\partial v_o}{\partial \theta} - w_o \right)$$

$$\hat{\theta}\hat{R} \left(\frac{\partial u_o}{\partial \theta} \right)$$

$$\hat{R} \left(\frac{\partial^2 u_1}{\partial \theta^2} + u_1 \right)$$

$$\hat{\phi} \left(\frac{\partial^2 v_1}{\partial \theta^2} - 2 \frac{\partial w_1}{\partial \theta} \right)$$

$$\hat{\theta} \left(\frac{\partial^2 w_1}{\partial \theta^2} + 2 \frac{\partial v_1}{\partial \theta} \right)$$

Since $\nabla \vec{\xi}$ and $\nabla^2 \vec{\xi}$ must be finite when $\phi = 0$, the above expressions (having $\frac{1}{\phi}$ as a multiplier) must all vanish. Thus we have

$$\frac{\partial w_o}{\partial \theta} = - v_o$$

$$\frac{\partial v_o}{\partial \theta} = w_o$$

$$\frac{\partial u_c}{\partial \theta} = 0$$

$$\frac{\partial^2 w_1}{\partial \theta^2} = - 2 \frac{\partial v_1}{\partial \theta}$$

$$\frac{\partial^2 v_1}{\partial \theta^2} = 2 \frac{\partial w_1}{\partial \theta}$$

$$\frac{\partial^2 u_1}{\partial \theta^2} = - u_1$$

The equations in (G23) are equivalent to these when $\phi = 0$. The equation

$$\frac{\partial \beta}{\partial \theta} = 0 \qquad (\phi = 0)$$

where

$$\beta = \int_{T_o}^{T} \alpha(T)dT$$

can be seen from the fact that at $\phi = 0$, there is only one point for all θ, so that β cannot vary with θ. (The same argument could have been applied to the quantity u. It could not be applied to w or v since these are functions which have extension in θ.)

III. DERIVATION OF THE FINITE-DIFFERENCE MODEL OF THE EQUATIONS

The finite-difference model of the basic equations is presented below. The difference analogs to the partial differential equations are constructed on a grid network as shown in Figure G2, for which α_1 - constant lines are ordered by the subscript i, α_2 = constant lines by the subscript j, α_3 = constant lines by the subscript k, and the intersection of grid lines (nodes) by the triple subscript i, j, k.

Note: The further evaluation of Equation (G35) may reduce the expression to a simple derivation process, but such a possibility will not be investigated since the expression has already been made determinate.

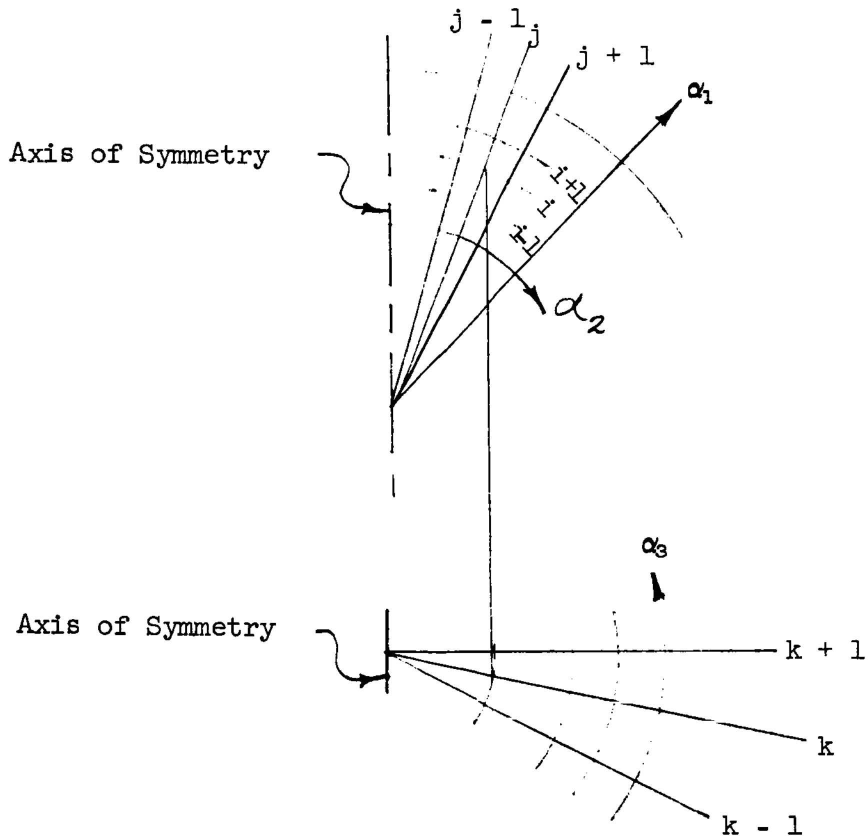


Fig. G2 - Grid Notation for Finite-Difference Formulation

For the general case, the grid spacing will be irregular and the increments in the vicinity of a node will be designated by the following notations:

$$h_{11} = (\alpha_1)_{i+1} - (\alpha_1)_i \qquad h_{21} = (\alpha_2)_{i+1} - (\alpha_2)_i \qquad h_{31} = (\alpha_3)_{i+1} - (\alpha_3)_i$$

$$h_{12} = (\alpha_1)_{i+2} - (\alpha_1)_i \qquad h_{22} = (\alpha_2)_{i+2} - (\alpha_2)_i \qquad h_{32} = (\alpha_3)_{i+2} - (\alpha_3)_i$$

$$h_{13} = (\alpha_1)_i - (\alpha_1)_{i-1} \qquad h_{23} = (\alpha_2)_i - (\alpha_2)_{i-1} \qquad h_{33} = (\alpha_3)_i - (\alpha_3)_{i-1}$$

$$h_{14} = (\alpha_1)_i - (\alpha_1)_{i-2} \qquad h_{24} = (\alpha_2)_i - (\alpha_2)_{i-2} \qquad h_{34} = (\alpha_3)_i - (\alpha_3)_{i-2}$$

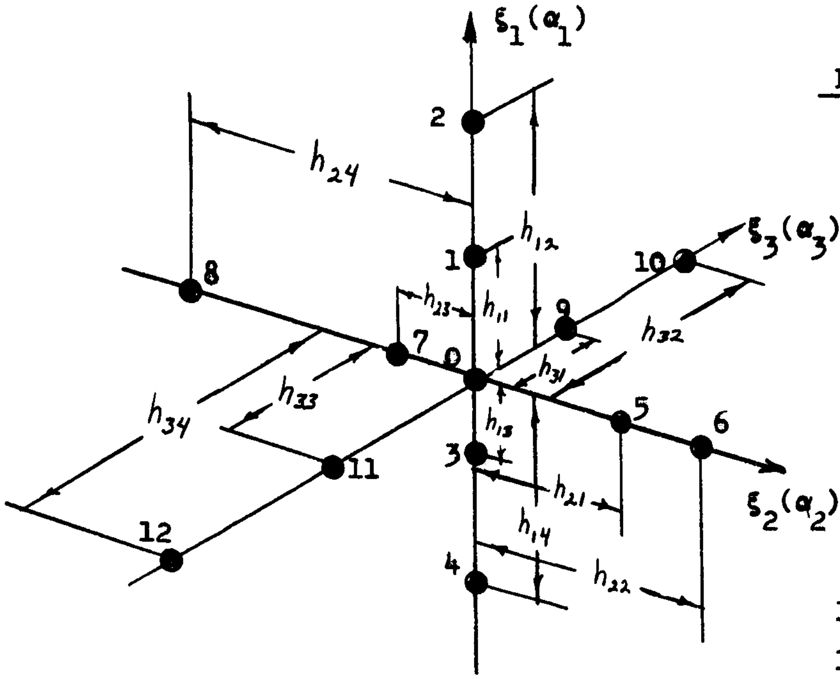
Let f $(\alpha_1, \alpha_2, \alpha_3)$ be any function of the coordinates such that it and its partial derivatives (up to any order required in the analysis) are continuous, and expand the function about the point i, j, k. Using a new coordinate system with origin at i, j, k and with ξ_1, ξ_2, ξ_3 directed along α_1, α_2, α_3, respectively, the function f (ξ_1, ξ_2, ξ_3) is written

$$\begin{aligned} f(\xi_1, \xi_2, \xi_3) = f_{i,j,k} &+ B_1\xi_1 + B_2\xi_2 + B_3\xi_3 + B_4\xi_1\xi_2 + B_5\xi_2\xi_3 \\ &+ B_6\xi_3\xi_1 + B_7\xi_1^{\,2} + B_8\xi_2^{\,2} + B_9\xi_3^{\,2} + B_{10}\xi_1\xi_2\xi_3 \\ &+ B_{11}\xi_1\xi_2^{\,2} + B_{12}\xi_1\xi_3^{\,2} + B_{13}\xi_1^{\,2}\xi_2 + B_{14}\xi_2\xi_3^{\,2} + \ldots \end{aligned} \tag{G36}$$

The first and second derivatives of f $(\alpha_1, \alpha_2, \alpha_3)$ with respect to α_1, α_2, α_3 are obtained from Equation (G36) according to

$$\left.\begin{aligned} \left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} = \left.\frac{\partial f}{\partial \xi_1}\right)_{(o,o,o)} = B_1, \quad \left.\frac{\partial^2 f}{\partial \alpha_1\,\partial \alpha_2}\right)_{i,j,k} = B_4, \quad \left.\frac{\partial^2 f}{\partial \alpha_1^{\,2}}\right)_{i,j,k} = 2\,B_7 \\ \left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} = \left.\frac{\partial f}{\partial \xi_2}\right)_{(o,o,o)} = B_2, \quad \left.\frac{\partial^2 f}{\partial \alpha_2\,\partial \alpha_3}\right)_{i,j,k} = B_5, \quad \left.\frac{\partial^2 f}{\partial \alpha_2^{\,2}}\right)_{i,j,k} = 2\,B_8 \\ \left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} = \left.\frac{\partial f}{\partial \xi_3}\right)_{(o,o,o)} = B_3, \quad \left.\frac{\partial^2 f}{\partial \alpha_3\,\partial \alpha_1}\right)_{i,j,k} = B_6, \quad \left.\frac{\partial^2 f}{\partial \alpha_3^{\,2}}\right)_{i,j,k} = 2\,B_9 \end{aligned}\right\} \tag{G37}$$

By considering the values of f (ξ_1, ξ_2, ξ_3) at the 12 nodes adjacent to i,j,k, the constants B_i are evaluated in terms of the function at these nodes and the grid spacings as shown in Figure G3.



Node	Coordinate
0	(i,j,k)
1	(i+1,j,k)
2	(i+2,j,k)
3	(i-1,j,k)
4	(i-2,j,k)
5	(i,j+1,k)
6	(i,j+2,k)
7	(i,j-1,k)
8	(i,j-2,k)
9	(i,j,k+1)
10	(i,j,k+2)
11	(i,j,k-1)
12	(i,j,k-2)

Fig. G3 - Coordinates of Irregular Mesh Intervals

Note that the grid spacing increments h_{ij} do not, in general, have the dimensions of length but have the dimensions of α_1, α_2, and α_3.

At points 1 and 3, Equation (G36) becomes

$$\left.\begin{aligned} f(h_{11,o,o}) &= f_{i,j,k} + B_1 h_{11} + B_7 h_{11}^2 \\ f(-h_{13,o,o}) &= f_{i,j,k} - B_1 h_{13} + B_7 h_{13}^2 \end{aligned}\right\} \qquad (G38)$$

where terms of higher order are deleted. Solving for B_1 and B_7 from Equation (G38) gives, for the first and second irregular central derivative with respect to α_1,

$$\left.\begin{aligned} \left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} &= \frac{h_{13}^2\, f_{i+1,j,k} + (h_{11}^2 - h_{13}^2)\, f_{i,j,k} - h_{11}^2\, f_{i-1,j,k}}{h_{11}h_{13}\,(h_{11} + h_{13})} \\ \left.\frac{\partial^2 f}{\partial \alpha_1^2}\right)_{i,j,k} &= 2\left[\frac{h_{13}f_{i+1,j,k} - (h_{11} + h_{13})\, f_{i,j,k} + h_{11}f_{i-1,j,k}}{h_{11}h_{13}\,(h_{11} + h_{13})}\right] \end{aligned}\right\} \qquad (G39)$$

Substituting $h_{11} = h_{13} = h_1$ into Equation (G39) gives for the first and second angular central derivatives with respect to α_1

$$\left.\begin{aligned} \left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} &= \frac{f_{i+1,j,k} - f_{i-1,j,k}}{2h_1} \\ \left.\frac{\partial^2 f}{\partial \alpha_1^2}\right)_{i,j,k} &= \frac{f_{i+1,j,k} - 2f_{i,j,k} + f_{i-1,j,k}}{h_1^2} \end{aligned}\right\} \qquad \text{(G40)}$$

By a similar procedure, the following first and second regular and irregular central derivatives are obtained with respect to the coordinates α_2 and α_3:

<u>First Regular Central Derivatives</u> $(h_2 = h_{21} = h_{23}, h_3 = h_{31} = h_{33})$

$$\left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} = \frac{f_{i,j+1,k} - f_{i,j-1,k}}{2h_2} \qquad \text{(G41)}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} = \frac{f_{i,j,k+1} - f_{i,j,k-1}}{2h_3} \qquad \text{(G42)}$$

<u>First Irregular Central Derivatives</u>

$$\left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} = \frac{h_{23}^2 f_{i,j+1,k} + (h_{21}^2 - h_{23}^2) f_{i,j,k} - h_{21}^2 f_{i,j-1,k}}{h_{21}h_{23}(h_{21} + h_{23})} \qquad \text{(G43)}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} = \frac{h_{33}^2 f_{i,j,k+1} + (h_{31}^2 - h_{33}^2) f_{i,j,k} - h_{31}^2 f_{i,j,k-1}}{h_{31}h_{33}(h_{31} + h_{33})} \qquad \text{(G44)}$$

<u>Second Regular Central Derivatives</u> $(h_2 = h_{21} = h_{23}, h_3 = h_{31} = h_{33})$

$$\left.\frac{\partial^2 f}{\partial \alpha_2^2}\right)_{i,j,k} = \frac{f_{i,j+1,k} - 2f_{i,j,k} + f_{i,j-1,k}}{h_2^2} \qquad \text{(G45)}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^{\,2}}\right)_{i,j,k} = \frac{f_{i,j,k+1} - 2f_{i,j,k} + f_{i,j,k-1}}{h_3^{\,2}} \tag{G46}$$

Second Irregular Central Derivatives

$$\left.\frac{\partial^2 f}{\partial \alpha_2^{\,2}}\right)_{i,j,k} = \frac{2\left[h_{23}f_{i,j+1,k} - (h_{21} + h_{23})\, f_{i,j,k} + h_{21}f_{i,j-1,k}\right]}{h_{21}h_{23}\,(h_{21} + h_{23})} \tag{G47}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^{\,2}}\right)_{i,j,k} = \frac{2\left[h_{33}f_{i,j,k+1} + (h_{31} + h_{33})\, f_{i,j,k} + h_{31}f_{i,j,k-1}\right]}{h_{31}h_{33}\,(h_{31} + h_{33})} \tag{G48}$$

Forward and Backward Derivatives

By applying the same procedure as above with respect to two nodes located either forward or backward from the origin (i,j,k), the first and second regular and irregular derivatives are obtained in terms of the function $f(\alpha_1, \alpha_2, \alpha_3)$ evaluated at these nodes. The results are summarized below for the three coordinate directions:

First Irregular Forward Derivatives

$$\left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} = \frac{-\,(h_{12}^{\,2} - h_{11}^{\,2})\, f_{i,j,k} + h_{12}^{\,2}\, f_{i+1,j,k} - h_{11}^{\,2}\, f_{i+2,j,k}}{h_{11}h_{12}\,(h_{12} - h_{11})} \tag{G49}$$

$$\left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} = \frac{-\,(h_{22}^{\,2} - h_{21}^{\,2})\, f_{i,j,k} + h_{22}^{\,2}\, f_{i,j+1,k} - h_{21}^{\,2}\, f_{i,j+2,k}}{h_{21}h_{22}\,(h_{22} - h_{21})} \tag{G50}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} = \frac{-\,(h_{32}^{\,2} - h_{31}^{\,2})\, f_{i,j,k} + h_{32}^{\,2}\, f_{i,j,k+1} - h_{31}^{\,2}\, f_{i,j,k+2}}{h_{31}h_{32}\,(h_{32} - h_{31})} \tag{G51}$$

First Regular Forward Derivatives

For equal grid spacings in each of the three coordinate directions, defined according to

$$\left.\begin{aligned} h_{11} &= h_{12}/2 \equiv h_1 \\ h_{21} &= h_{22}/2 \equiv h_2 \\ h_{31} &= h_{32}/2 \equiv h_3 \end{aligned}\right\} \tag{G52}$$

Equations (G49) through (G51) reduce to

$$\left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} = \frac{-3f_{i,j,k} + 4f_{i+1,j,k} - f_{i+2,j,k}}{2h_1} \tag{G53}$$

$$\left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} = \frac{-3f_{i,j,k} + 4f_{i,j+1,k} - f_{i,j+2,k}}{2h_2} \tag{G54}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} = \frac{-3f_{i,j,k} + 4f_{i,j,k+1} - f_{i,j,k+2}}{2h_3} \tag{G55}$$

Second Irregular Forward Derivatives

$$\left.\frac{\partial^2 f}{\partial \alpha_1{}^2}\right)_{i,j,k} = 2\left[\frac{-h_{12}f_{i+1,j,k} + (h_{12} - h_{11})\, f_{i,j,k} + h_{11}f_{i+2,j,k}}{h_{11}h_{12}\,(h_{12} - h_{11})}\right] \tag{G56}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2{}^2}\right)_{i,j,k} = 2\left[\frac{-h_{22}f_{i,j+1,k} + (h_{22} - h_{21})\, f_{i,j,k} + h_{21}f_{i,j+2,k}}{h_{21}h_{22}\,(h_{22} - h_{21})}\right] \tag{G57}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3{}^2}\right)_{i,j,k} = 2\left[\frac{-h_{32}f_{i,j,k+1} + (h_{32} - h_{31})\, f_{i,j,k} + h_{31}f_{i,j,k+2}}{h_{31}h_{32}\,(h_{32} - h_{31})}\right] \tag{G58}$$

Second Regular Forward Derivatives

With equal grid spacing, according to Equation (G52), Equations (G56) through (G58) reduce to

$$\left.\frac{\partial^2 f}{\partial \alpha_1{}^2}\right)_{i,j,k} = \frac{-2f_{i+1,j,k} + f_{i,j,k} + f_{i+2,j,k}}{h_1{}^2} \tag{G59}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2{}^2}\right)_{i,j,k} = \frac{-2f_{i,j+1,k} + f_{i,j,k} + f_{i,j+2,k}}{h_2{}^2} \tag{G60}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3{}^2}\right)_{i,j,k} = \frac{-2f_{i,j,k+1} + f_{i,j,k} + f_{i,j,k+2}}{h_3{}^2} \tag{G61}$$

First Irregular Backward Derivatives

$$\left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} = \frac{h_{13}{}^2 f_{i-2,j,k} + (h_{14}{}^2 - h_{13}{}^2)\, f_{i,j,k} - h_{14}{}^2 f_{i-1,j,k}}{h_{13} h_{14} (h_{14} - h_{13})} \tag{G62}$$

$$\left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} = \frac{h_{23}{}^2 f_{i,j-2,k} + (h_{24}{}^2 - h_{23}{}^2)\, f_{i,j,k} - h_{24}{}^2 f_{i,j-1,k}}{h_{23} h_{24} (h_{24} - h_{23})} \tag{G63}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} = \frac{h_{33}{}^2 f_{i,j,k-2} + (h_{34}{}^2 - h_{33}{}^2)\, f_{i,j,k} - h_{34}{}^2 f_{i,j,k-1}}{h_{33} h_{34} (h_{34} - h_{33})} \tag{G64}$$

First Regular Backward Derivatives ($h_{13} = h_{14}/2 \equiv h_1$, etc.)

$$\left.\frac{\partial f}{\partial \alpha_1}\right)_{i,j,k} = \frac{f_{i-2,j,k} + 3f_{i,j,k} - 4f_{i-1,j,k}}{2h_1} \tag{G65}$$

$$\left.\frac{\partial f}{\partial \alpha_2}\right)_{i,j,k} = \frac{f_{i,j-2,k} + 3f_{i,j,k} - 4f_{i,j-1,k}}{2h_2} \tag{G66}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right)_{i,j,k} = \frac{f_{i,j,k-2} + 3f_{i,j,k} - 4f_{i,j,k-1}}{2h_3} \tag{G67}$$

Second Irregular Backward Derivatives

$$\left.\frac{\partial^2 f}{\partial \alpha_1{}^2}\right)_{i,j,k} = 2\left[\frac{h_{13} f_{i-2,j,k} + (h_{14} - h_{13})\, f_{i,j,k} - h_{14} f_{i-1,j,k}}{h_{13} h_{14} (h_{14} - h_{13})}\right] \tag{G68}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2{}^2}\right)_{i,j,k} = 2\left[\frac{h_{23} f_{i,j-2,k} + (h_{24} - h_{23})\, f_{i,j,k} - h_{24} f_{i,j-1,k}}{h_{23} h_{24} (h_{24} - h_{23})}\right] \tag{G69}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3{}^2}\right)_{i,j,k} = 2\left[\frac{h_{33} f_{i,j,k-2} + (h_{34} - h_{33})\, f_{i,j,k} - h_{34} f_{i,j,k-1}}{h_{33} h_{34} (h_{34} - h_{33})}\right] \tag{G70}$$

Second Regular Backward Derivatives

$$\left.\frac{\partial^2 f}{\partial \alpha_1{}^2}\right)_{i,j,k} = \frac{f_{i-2,j,k} + f_{i,j,k} - 2f_{i-1,j,k}}{h_1{}^2} \tag{G71}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2{}^2}\right)_{i,j,k} = \frac{f_{1,j-2,k} + f_{i,j,k} - 2f_{i,j-1,k}}{h_2{}^2} \tag{G72}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3{}^2}\right)_{i,j,k} = \frac{f_{i,j,k-2} + f_{i,j,k} - 2f_{i,j,k-1}}{h_3{}^2} \tag{G73}$$

Mixed Derivatives

It can be shown from Equation (G36) that mixed derivatives require values of the function at any six nodes in the vicinity of the point under consideration. Figure G4 shows various combinations of mixed derivatives with respect to the coordinate axes α_1 and α_2. It is noted that the mixed central derivatives involve the four corner nodes as well as two adjacent nodes in either of the two coordinate directions. The various combinations shown in Figure G4 are summarized below for the coordinate directions α_1 and α_2:

Second Mixed Irregular Central Derivative with Respect to α_1 and α_2

a) $$\left.\frac{\partial^2 f}{\partial \alpha_1\, \partial \alpha_2}\right)_{i,j,k} = \frac{1}{h_{21} h_{23} (h_{11} + h_{13}) (h_{21} + h_{23})} \Big[h_{23}{}^2 \left(f_{i+1,j+1,k} - f_{i-1,j+1,k}\right) - (h_{23}{}^2 - h_{21}{}^2) \left(f_{i+1,j,k} - f_{i-1,j,k}\right) - h_{21}{}^2 \left(f_{i+1,j-1,k} - f_{i-1,j-1,k}\right)\Big] \tag{G74}$$

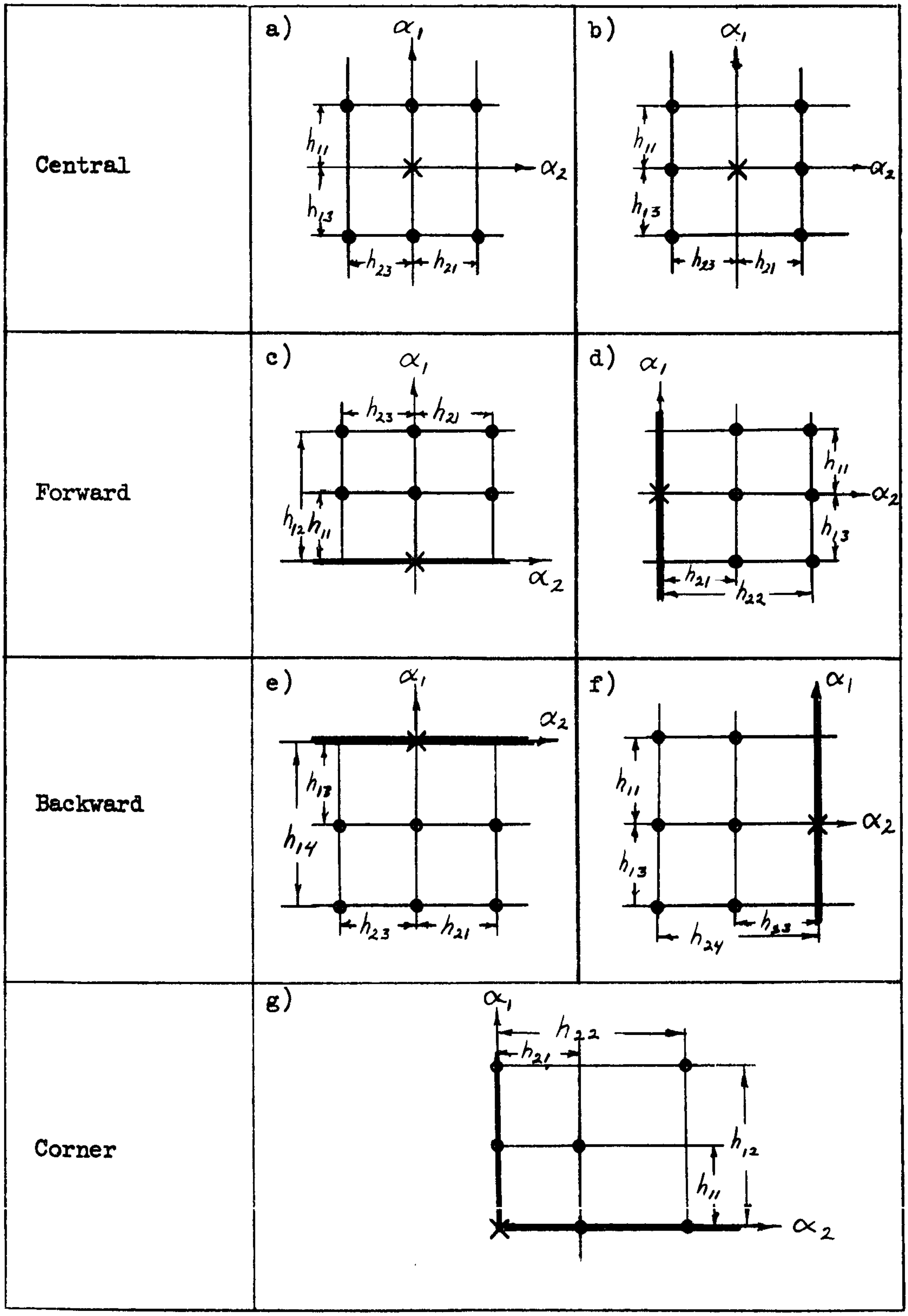


Fig. G4 - Irregular Mesh Intervals for Mixed Central, Forward Backward, and Corner Derivatives

b) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right)_{i,j,k} = \frac{1}{h_{11}h13\,(h_{11}+h_{13})\,(h_{21}+h_{23})}\Big[h_{13}^{\;2}\,\big(f_{i+1,j+1,k} - f_{i+1,j-1,k}\big) - (h_{13}^{\;2} - h_{11}^{\;2})\,\big(f_{i,j+1,k} - f_{i,j-1,k}\big) - h_{11}^{\;2}\,\big(f_{i-1,j+1,k} - f_{i-1,j-1,k}\big)\Big] \quad \text{(G75)}$$

Second Mixed Irregular Forward Derivative with Respect to α_1 and α_2

c) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right)_{i,j,k} = \frac{1}{h_{21}h_{23}\,(h_{11}-h_{12})\,(h_{21}+h_{23})}\Big[h_{23}^{\;2}\,\big(f_{i+1,j+1,k} - f_{i+1,j,k} - f_{i+2,j+1,k} + f_{i+2,j,k}\big) - h_{21}^{\;2}\,\big(f_{i+1,j-1,k} - f_{i+1,j,k} - f_{i+2,j-1,k} + f_{i+2,j,k}\big)\Big] \quad \text{(G76)}$$

d) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right)_{i,j,k} = \frac{1}{h_{11}h_{13}\,(h_{21}-h_{22})\,(h_{11}+h_{13})}\Big[h_{13}^{\;2}\,\big(f_{i+1,j+1,k} - f_{i,j+1,k} - f_{i+1,j+2,k} + f_{i,j+2,k}\big) - h_{11}^{\;2}\,\big(f_{i-1,j+1,k} - f_{i,j+1,k} - f_{i-1,j+2,k} + f_{i,j+2,k}\big)\Big] \quad \text{(G77)}$$

Second Mixed Irregular Backward Derivative with Respect to α_1 and α_2

e) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right)_{i,j,k} = \frac{1}{h_{21}h_{23}\,(h_{13}-h_{14})\,(h_{21}+h_{23})}\Big[h_{23}^{\;2}\,\big(f_{i-1,j+1,k} - f_{i-2,j+1,k} + f_{i-2,j,k} - f_{i-1,j,k}\big) - h_{21}^{\;2}\,\big(f_{i-1,j-1,k} - f_{i-1,j,k} - f_{i-2,j-1,k} + f_{i-2,j,k}\big)\Big]$$

f) $$\left.\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2}\right)_{i,j,k} = \frac{-1}{h_{11}h_{13}\left(h_{23}-h_{24}\right)\left(h_{11}+h_{13}\right)}\Big[h_{13}^{\;2}\left(f_{i+1,j-1,k} - f_{i+1,j-2,k} + f_{i,j-2,k} - f_{i,j-1,k}\right) - h_{11}^{\;2}\left(f_{i-1,j-1,k} - f_{i,j-1,k} - f_{i-1,j-2,k} + f_{i,j-2,k}\right)\Big] \qquad (G79)$$

Second Mixed Irregular Corner Derivative with Respect to α_1 and α_2

g) $$\frac{\partial^2 f}{\partial\alpha_1\,\partial\alpha_2} = \frac{1}{h_{11}h_{12}h_{21}h_{22}\left(h_{22}-h_{21}\right)}\Big[h_{12}h_{22}^{\;2}\left(f_{i+1,j+1,k} - f_{i+1,j,k} - f_{i,j+1,k} + f_{i,j,k}\right) - h_{11}h_{21}^{\;2}\left(f_{i+2,j+2,k} - f_{i+2,j,k} - f_{i,j+2,k} + f_{i,j,k}\right)\Big] \qquad (G80)$$

Second Mixed Regular Derivatives

All of the above results can be reduced to regular derivatives with respect to either α_1, α_2 or both coordinates by making the substitutions

$$h_{11} = h_{13} = \frac{h_{12}}{2} = \frac{h_{14}}{2} \equiv h_1 \qquad (G81)$$

$$h_{21} = h_{23} = \frac{h_{22}}{2} = \frac{h_{24}}{2} \equiv h_2 \qquad (G82)$$

$$h_1 = h_2 = h \qquad (G83)$$

The various derivatives are summarized below for the case in which all grid spacings are equal (i.e., $h_1 = h_2$).

Second Mixed Regular Central Derivative with Respect to α_1 and α_2

$$\text{a), b)} \quad \left.\frac{\partial^2 f}{\partial\alpha_1 \, \partial\alpha_2}\right)_{i,j,k} = \frac{1}{4\,h^2}\left(f_{i+1,j+1,k} - f_{i+1,j-1,k} - f_{i-1,j+1,k} + f_{i-1,j-1,k}\right) \tag{G84}$$

Second Mixed Regular Forward Derivative with Respect to α_1 and α_2

$$\text{c)} \quad \left.\frac{\partial^2 f}{\partial\alpha_1 \, \partial\alpha_2}\right)_{i,j,k} = \frac{-1}{2\,h_2}\left(f_{i+1,j+1,k} - f_{i+2,j+1,k} - f_{i+1,j-1,k} + f_{i+2,j-1,k}\right) \tag{G85}$$

$$\text{d)} \quad \left.\frac{\partial^2 f}{\partial\alpha_1 \, \partial\alpha_2}\right)_{i,j,k} = \frac{-1}{2\,h^2}\left(f_{i+1,j+1,k} - f_{i+1,j+2,k} - f_{i-1,j+1,k} + f_{i-1,j+2,k}\right) \tag{G86}$$

Second Mixed Regular Backward Derivative with Respect to α_1 and α_2

$$\text{e)} \quad \left.\frac{\partial^2 f}{\partial\alpha_1 \, \partial\alpha_2}\right)_{i,j,k} = \frac{1}{2\,h^2}\left(f_{i-1,j+1,k} - f_{i-2,j+1,k} - f_{i-1,j-1,k} + f_{i-2,j-1,k}\right) \tag{G87}$$

$$\text{f)} \quad \left.\frac{\partial^2 f}{\partial\alpha_1 \, \partial\alpha_2}\right)_{i,j,k} = \frac{1}{2\,h^2}\left(f_{i+1,j-1,k} - f_{i+1,j-2,k} - f_{i-1,j-1,k} + f_{i-1,j-2,k}\right) \tag{G88}$$

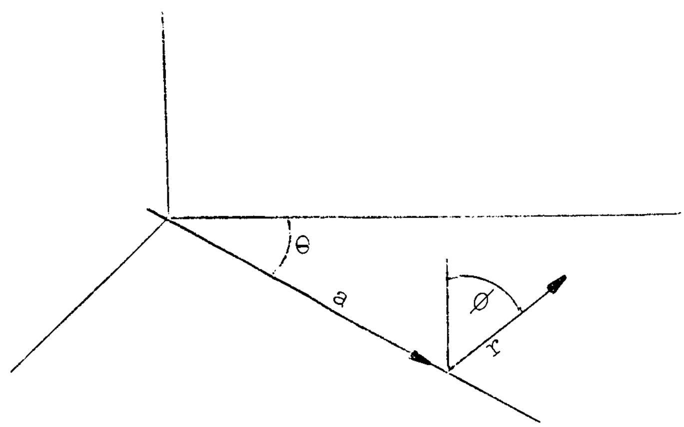
Second Mixed Regular Corner Derivative with Respect to α_1 and α_2

$$\text{g)} \quad \left.\frac{\partial^2 f}{\partial\alpha_1 \, \partial\alpha_2}\right)_{i,j,k} = \frac{1}{4\,h^2}\Big[8\left(f_{i+1,j+1,k} - f_{i+1,j,k} - f_{i,j+1,k} + f_{i,j,k}\right) - \left(f_{i+2,j+2,k} - f_{i+2,j,k} - f_{i,j+2,k} + f_{i,j,k}\right)\Big] \tag{G89}$$

IV. DEVELOPMENT OF THIN SHELL MODEL FOR THIN LAYER AND ITS FINITE-DIFFERENCE EQUIVALENT

A. THE THIN SHELL IN SPHERICAL AND TOROIDAL COORDINATES

The thin-shell analysis used here has been derived from a rigorous, direct expansion in appropriate powers of z (the distance from shell center surface). The equations are derived in the toroidal system since the spherical case can be obtained from these by putting a = 0.



Strain-displacement relations are shown in Equations (G90) through (G95).

$$\epsilon_{rr} = \frac{\partial u}{\partial r} \tag{G90}$$

$$\epsilon_{r\phi} = \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \phi}\right) \tag{G91}$$

$$\epsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial w}{\partial r} - \frac{w \sin \phi}{a + r \sin \phi} + \frac{1}{a + r \sin \phi}\frac{\partial u}{\partial \theta}\right) \tag{G92}$$

$$\epsilon_{\phi\phi} = \frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{u}{r} \tag{G93}$$

$$\epsilon_{\theta\theta} = \frac{1}{a + r \sin \phi}\left(\frac{\partial w}{\partial \theta} + u \sin \phi + v \cos \phi\right) \tag{G94}$$

$$\epsilon_{\phi\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial \phi} + \frac{1}{a + r \sin \phi}\frac{\partial v}{\partial \theta} - \frac{w \cos \phi}{a + r \sin \phi}\right) \tag{G95}$$

The stress-strain relations are as follows:

$$\epsilon_{rr} = \frac{\sigma_{rr}}{E} - \frac{\nu}{E}\left(\sigma_{\phi\phi} + \sigma_{\theta\theta}\right) + \alpha T \tag{G96}$$

$$\epsilon_{r\phi} = \frac{\sigma_{r\phi}}{2G} \tag{G97}$$

$$\epsilon_{r\theta} = \frac{\sigma_{r\theta}}{2G} \tag{G98}$$

$$\epsilon_{\phi\theta} = \frac{\sigma_{\phi\theta}}{2G} \tag{G99}$$

$$\epsilon_{\phi\phi} = \frac{\sigma_{\phi\phi}}{E} - \frac{\nu}{E}\left(\sigma_{rr} + \sigma_{\theta\theta}\right) + \alpha T \tag{G100}$$

$$\epsilon_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - \frac{\nu}{E}\left(\sigma_{rr} + \sigma_{\phi\phi}\right) + \alpha T \tag{G101}$$

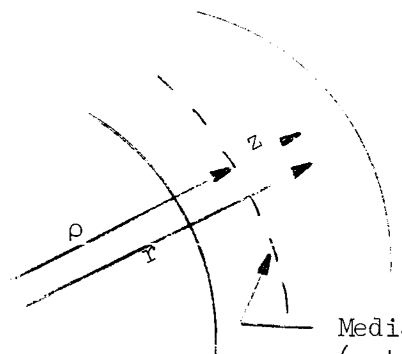
The first-order expansion* is

$$r = \rho + z$$

Thus,

$$\frac{1}{r} = \frac{1}{\rho} - \frac{z}{\rho^2}$$

$$u = u_o + \frac{z}{\rho} u_1$$



Median surface (not necessarily center)

etc., including ν, E, G, α, T since these are variables.

From Equations (G90), (G91), and (G92), we have

$$\epsilon_{rr_o} = \frac{u_1}{\rho} \tag{G102}$$

---

*Wherever terms are to be differentiated with respect to r and then combined with first-order terms, they must be carried to second-order in $\frac{z}{\rho}$, since the differentiation reduces the order by one.

$$\epsilon_{r\phi_o} = \frac{v_1}{\rho} - \frac{v_o}{\rho} + \frac{1}{\rho} \frac{\partial u_o}{\partial \phi} \tag{G103}$$

$$\epsilon_{r\theta_o} = \frac{w_1}{\rho} - \frac{w_o \sin \phi}{s} - \frac{1}{s} \frac{\partial u_o}{\partial \theta} \qquad (s = a + \sin \phi) \tag{G104}$$

Put $\epsilon_{\phi\phi} = \epsilon_{\phi\phi_o} + \frac{z}{\rho} \epsilon_{\phi\phi_1}$ etc.

From Equations (G93), (G94), and (G95)

$$\epsilon_{\phi\phi_o} = \frac{1}{\rho} \frac{\partial v_o}{\partial \phi} + \frac{u_o}{\rho} \tag{G105}$$

$$\epsilon_{\phi\phi_1} = \frac{1}{\rho} \frac{\partial v_1}{\partial \phi} - \frac{1}{\rho} \frac{\partial v_o}{\partial \phi} + \frac{u_1}{\rho} - \frac{u_o}{\rho} \tag{G106}$$

$$\epsilon_{\theta\theta_o} = \frac{1}{s} \left(- \frac{\partial w_o}{\partial \theta} + u_o \sin \phi + v_o \cos \phi \right) \tag{G107}$$

$$\epsilon_{\theta\theta_1} = \frac{1}{s} \left(- \frac{\partial w_1}{\partial \theta} + u_1 \sin \phi + v_1 \cos \phi \right) - \frac{\rho \sin \phi}{s^2} \left(- \frac{\partial w_o}{\partial \theta} + u_o \sin \phi + v_o \cos \phi \right) \tag{G108}$$

$$\epsilon_{\phi\theta_o} = \frac{1}{\rho} \frac{\partial w_o}{\partial \phi} - \frac{w_o \cos \phi}{s} - \frac{1}{s} \frac{\partial v_o}{\partial \theta} \tag{G109}$$

$$\epsilon_{\phi\theta_1} = \frac{1}{\rho} \left(\frac{\partial w_1}{\partial \phi} - \frac{\partial w_o}{\partial \phi} \right) - \frac{1}{s} \left(\frac{\partial v_1}{\partial \theta} + w_1 \cos \theta \right) + \frac{\rho \sin \phi}{s^2} \left(\frac{\partial v_o}{\partial \theta} + w_o \cos \phi \right) \tag{G110}$$

From Equations (G96), (G97), (G98), (G102), (G103), and (G104) we get

$$u_1 = \rho \left[\frac{\sigma_{rr_o}}{E_o} - \frac{\nu_o}{E_o} \left(\sigma_{\phi\phi_o} + \sigma_{\theta\theta_o} \right) + \alpha_o T_o \right] \tag{G111}$$

$$v_1 = v_o - \frac{\partial u_o}{\partial \phi} + \rho \frac{\sigma_{r\phi_o}}{G_o} \tag{G112}$$

$$w_1 = \frac{\rho}{s}\left(w_o \sin \phi + \frac{\partial u_o}{\partial \theta}\right) + \rho \frac{\sigma_{r\theta_o}}{G_o} \tag{G113}$$

For boundary stresses, second-order terms are needed because of later differentiations.

$$\sigma_{rr} = \sigma_{rr_o} + \frac{z}{\rho}\sigma_{rr_1} + \left(\frac{z}{\rho}\right)^2 \sigma_{rr_2}$$

At upper and lower surfaces $\left(\pm \frac{h}{2}\right)$, we have the pressures

$$-p_2 = \sigma_{rr_o} + \frac{h}{2\rho}\sigma_{rr_1} + \left(\frac{h}{2\rho}\right)^2 \sigma_{rr_2} \qquad \text{(for upper surface)}$$

$$-p_1 = \sigma_{rr_o} - \frac{h}{2\rho}\sigma_{rr_1} + \left(\frac{h}{2\rho}\right)^2 \sigma_{rr_2} \qquad \text{(for lower surface)}$$

Thus,

$$\sigma_{rr_1} = (p_1 - p_2)\frac{\rho}{h} \tag{G114}$$

$$\sigma_{rr_o} = -\left(\frac{p_1 + p_2}{2}\right) - \left(\frac{h}{2\rho}\right)^2 \sigma_{rr_2} \tag{G115}$$

For $\sigma_{r\phi}$, $\sigma_{r\theta}$, in terms of the upper and lower shear stresses, it is seen that

$$\sigma_{r\phi_1} = \left(\tau_{\phi_1} - \tau_{\phi_2}\right)\frac{\rho}{h} \tag{G116}$$

$$\sigma_{r\theta_1} = \left(\tau_{\theta_1} - \tau_{\theta_2}\right)\frac{\rho}{h} \tag{G117}$$

$$\sigma_{r\phi_o} = -\frac{\left(\tau_{\phi_1} + \tau_{\phi_2}\right)}{2} - \left(\frac{h}{2\rho}\right)^2 \sigma_{r\phi_2} \tag{G118}$$

$$\sigma_{r\theta_o} = -\frac{\left(\tau_{\theta_1} + \tau_{\theta_2}\right)}{2} - \left(\frac{h}{2\rho}\right)^2 \sigma_{r\theta_2} \tag{G119}$$

To put $\sigma_{\phi\phi_o} + \sigma_{\theta\theta_o}$ in Equation (G111) in terms of displacements, use Equations (G100) and (G101); thus

$$\epsilon_{\phi\phi_o} + \epsilon_{\theta\theta_o} = \frac{1 - \nu_o}{E_o} \left(\sigma_{\phi\phi_o} + \sigma_{\theta\theta_o} \right) - \frac{2\nu_o}{E_o} \sigma_{rr_o} + 2\alpha_o T_o$$

Using Equations (G105) and (G107), we have then

$$\frac{1 - \nu_o}{E_o} \left(\sigma_{\phi\phi_o} + \sigma_{\theta\theta_o} \right) = \frac{1}{\rho} \frac{\partial v_o}{\partial \phi} + \frac{u_o}{\rho} - \frac{1}{s} \frac{\partial w_o}{\partial \theta} + \frac{u_o \sin \phi + v_o \cos \phi}{s} + \frac{2\nu_o}{E_o} \sigma_{rr_o} - 2\alpha_o T_o \tag{G120}$$

Then by putting Equation (G120) into Equation (G111), we get

$$u_1 = - \frac{\nu_o}{1 - \nu_o} \left[\left(\frac{\partial v_o}{\partial \phi} + u_o \right) + \frac{\rho}{s} \left(- \frac{\partial w_o}{\partial \theta} + u_o \sin \phi + v_o \cos \phi \right) \right] + \frac{\rho}{E_o} \frac{(1 - 2\nu_o)(1 + \nu_o)}{(1 - \nu_o)} \sigma_{rr_o} + \rho \left(\frac{1 + \nu_o}{1 - \nu_o} \right) \alpha_o T_o \tag{G121}$$

Equations of equilibrium are as follows:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\phi}}{\partial \phi} - \frac{1}{a + r \sin \phi} \frac{\partial \sigma_{r\theta}}{\partial \theta} = 0$$

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\phi\phi}}{\partial \phi} - \frac{1}{a + r \sin \phi} \frac{\partial \sigma_{\phi\theta}}{\partial \theta} = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\phi\theta}}{\partial \phi} - \frac{1}{a + r \sin \phi} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0$$

These result in six equations in zero-order and first-order terms; thus

$$\sigma_{rr_1} + \frac{\partial \sigma_{r\phi_o}}{\partial \phi} - \frac{\rho}{s} \frac{\partial \sigma_{r\theta_o}}{\partial \theta} = 0 \tag{G122}$$

$$\sigma_{r\phi_1} + \frac{\partial \sigma_{\phi\phi_o}}{\partial \phi} - \frac{\rho}{s} \frac{\partial \sigma_{\phi\theta_o}}{\partial \theta} = 0 \tag{G123}$$

$$\sigma_{r\theta_1} + \frac{\partial \sigma_{\phi\theta_o}}{\partial \phi} - \frac{\rho}{s} \frac{\partial \sigma_{\theta\theta_o}}{\partial \theta} = 0 \tag{G124}$$

$$2\,\sigma_{rr_2} + \frac{\partial \sigma_{r\phi_1}}{\partial \phi} - \frac{\rho}{s}\,\frac{\partial \sigma_{r\theta_1}}{\partial \theta} - \frac{\partial \sigma_{r\phi_o}}{\partial \phi} + \frac{\rho^2 \sin \phi}{s^2}\,\frac{\partial \sigma_{r\theta_o}}{\partial \theta} = 0 \qquad \text{(G125)}$$

$$2\,\sigma_{r\phi_2} + \frac{\partial \sigma_{\phi\phi_1}}{\partial \phi} - \frac{\rho}{s}\,\frac{\partial \sigma_{\phi\theta_1}}{\partial \theta} - \frac{\partial \sigma_{\phi\phi_o}}{\partial \phi} + \frac{\rho^2 \sin \phi}{s^2}\,\frac{\partial \sigma_{\phi\theta_o}}{\partial \theta} = 0 \qquad \text{(G126)}$$

$$2\,\sigma_{r\theta_2} + \frac{\partial \sigma_{\phi\theta_1}}{\partial \phi} - \frac{\rho}{s}\,\frac{\partial \sigma_{\theta\theta_1}}{\partial \theta} - \frac{\partial \sigma_{\phi\theta_o}}{\partial \phi} + \frac{\rho^2 \sin \phi}{s^2}\,\frac{\partial \sigma_{\theta\theta_o}}{\partial \theta} = 0 \qquad \text{(G127)}$$

Equations (G99), (G100), and (G101) give $\sigma_{\phi\phi}$, $\sigma_{\phi\theta}$, $\sigma_{\theta\theta}$ in terms of the strains; thus

$$\sigma_{\phi\phi} = \frac{E}{1 - \nu^2}\left(\epsilon_{\phi\phi} + \nu\,\epsilon_{\theta\theta}\right) - \frac{E\alpha T}{1 - \nu} + \frac{\nu}{1 - \nu}\,\sigma_{rr} \qquad \text{(G128)}$$

$$\sigma_{\theta\theta} = \frac{E}{1 - \nu^2}\left(\epsilon_{\theta\theta} + \nu\,\epsilon_{\phi\phi}\right) - \frac{E\alpha T}{1 - \nu} + \frac{\nu}{1 - \nu}\,\sigma_{rr} \qquad \text{(G129)}$$

$$\sigma_{\phi\theta} = \frac{E}{1 + \nu}\,\epsilon_{\phi\theta} \qquad \text{(G130)}$$

These relationships give six equations for the quantities $\sigma_{\phi\phi_o}$, $\sigma_{\phi\phi_1}$, etc. The zero-order expressions are

$\sigma_{\phi\phi_o}$ = Equation (G128) with all quantities given zero subscript (e.g., E_o, $\epsilon_{\phi\phi_o}$, α_o, etc.) (G131)

$\sigma_{\theta\theta_o}$ = Equation (G129) with all quantities given zero subscript (e.g., E_o, $\epsilon_{\phi\phi_o}$, α_o, etc.) (G132)

$\sigma_{\phi\theta_o}$ = Equation (G130) with all quantities given zero subscript (e.g., E_o, $\epsilon_{\phi\phi_o}$, α_o, etc.) (G133)

The first-order expressions are much more complicated. Thus,

$$\sigma_{\theta\theta_1} = \frac{E_o}{1 - \nu_o^2} \epsilon_{\theta\theta_1} + \frac{E_o \nu_o}{1 - \nu_o^2} \epsilon_{\phi\phi_1} + \left[\frac{E_1}{1 - \nu_o^2} + \frac{2 E_o \nu_o \nu_1}{(1 - \nu_o^2)^2}\right] \epsilon_{\theta\theta_o}$$

$$+ \left[\frac{E_1 \nu_o}{1 - \nu_o^2} + \frac{E_o \nu_1 (1 + \nu_o^2)}{(1 - \nu_o^2)^2}\right] \epsilon_{\phi\phi_o}$$

$$- \left[\frac{E_o \alpha_o T_1}{1 - \nu_o} + \frac{E_o \alpha_1 T_o}{1 - \nu_o} + \frac{E_1 \alpha_o T_o}{1 - \nu_o} + \frac{E_o \alpha_o T_o \nu_1}{(1 - \nu_o)^2}\right]$$

$$+ \left[\frac{\nu_o}{1 - \nu_o}\right] \sigma_{rr_1} + \frac{\nu_1}{(1 - \nu_o)^2} \sigma_{rr_o} \tag{G134}$$

$$\sigma_{\phi\phi_1} = \frac{E_o}{1 - \nu_o^2} \epsilon_{\phi\phi_1} + \frac{E_o \nu_o}{1 - \nu_o^2} \epsilon_{\theta\theta_1} + \left[\frac{E_1}{1 - \nu_o^2} + \frac{2 E_o \nu_o \nu_1}{(1 - \nu_o^2)^2}\right] \epsilon_{\phi\phi_o}$$

$$+ \left[\frac{E_1 \nu_o}{1 - \nu_o^2} + \frac{E_o \nu_1 (1 + \nu_o^2}{(1 - \nu_o^2)^2}\right] \epsilon_{\theta\theta_o}$$

$$- \left[\frac{E_o \alpha_o T_1}{1 - \nu_o} + \frac{E_o \alpha_1 T_o}{1 - \nu_o} + \frac{E_1 \alpha_o T_o}{1 - \nu_o} + \frac{E_o \alpha_o T_o \nu_1}{(1 - \nu_o)^2}\right]$$

$$+ \left[\frac{\nu_o}{1 - \nu_o}\right] \sigma_{rr_1} + \frac{\nu_1}{(1 - \nu_o)^2} \sigma_{rr_o} \tag{G135}$$

$$\sigma_{\phi\theta_1} = \frac{E_o}{1 + \nu_o} \epsilon_{\phi\theta_1} + \left[\frac{E_o}{1 + \nu_o} - \frac{E_o \nu_1}{(1 + \nu_o)^2}\right] \epsilon_{\phi\theta_o} \tag{G136}$$

Equations (G112), (G113), and (G121) are substituted into Equations (G106), (G108), and (G110); then Equations (G105) through (G110) are substituted into Equations (G131) through (G136). The resulting equations give $\sigma_{\theta\theta_o}$, $\sigma_{\theta\theta_1}$, $\sigma_{\phi\phi_o}$, $\sigma_{\phi\phi_1}$, $\sigma_{\phi\theta_o}$, $\sigma_{\phi\theta_1}$ in terms of u_o, u_1, v_o, v_1, w_o, w_1, σ_{rr_o}, σ_{rr_1}, $\sigma_{r\theta_o}$, $\sigma_{r\phi_o}$.

Thus,

$\sigma_{\phi\phi_o}$ is obtained from Equations (G105), (G107), and (G131) (G137)

$\sigma_{\phi\phi_1}$ is obtained from Equations (G105), (G106), (G107), (G108), (G112), (G113), (G121), and (G135). (G138)

$\sigma_{\theta\theta_o}$ is obtained from Equations (G105), (G107), and (G132) (G139)

$\sigma_{\theta\theta_1}$ is obtained from Equations (G134), (G105), (G106), (G107), (G108), (G112), (G113), and (G121) (G140)

$\sigma_{\phi\theta_o}$ is obtained from Equations (G109) and (G133) (G141)

$\sigma_{\phi\theta_1}$ is obtained from Equations (G109), (G110), (G112), (G113), and (G136)

When Equations (G137) through (G142) are substituted into Equations (G122) through (G127), the result is <u>six</u> differential equations in the six quantities u_o, v_o, w_o, σ_{rr_2}, $\sigma_{r\phi_2}$, $\sigma_{r\theta_2}$ [σ_{rr_1}, σ_{rr_o}, etc. are already given in terms of surface stresses, Equations (G114) through (G119)]. The corresponding equations for the spherical system are obtained by letting a = 0.

The first-order expansions of the tangential stresses are

$$\sigma_{\phi\phi} = \sigma_{\phi\phi_o} + \left(\frac{z}{\rho}\right) \sigma_{\phi\phi_1} \tag{G143}$$

$$\sigma_{\theta\theta} = \sigma_{\theta\theta_o} + \left(\frac{z}{\rho}\right) \sigma_{\theta\theta_1} \tag{G144}$$

$$\sigma_{\phi\theta} = \sigma_{\phi\theta_o} + \left(\frac{z}{\rho}\right) \sigma_{\phi\theta_1} \tag{G145}$$

where $\sigma_{\phi\phi_o}$, $\sigma_{\phi\phi_1}$, $\sigma_{\theta\theta_o}$, $\sigma_{\theta\theta_1}$, $\sigma_{\phi\theta_o}$, and $\sigma_{\phi\theta_1}$ have been derived in Equations (G137) through (G142). Substituting these equations into Equations (G143), (G144), and (G145), the stress equations in toroidal coordinates are obtained in terms of u_o, v_o, w_o, and σ_{rr_2}.

$$
\left.
\begin{aligned}
\sigma_{\phi\phi} = {} & C_1 \left[\frac{1}{\rho}\right] \frac{\partial v_o}{\partial \phi} - C_1 \left[\frac{\nu_o}{s}\right] \frac{\partial w_o}{\partial \theta} + C_1 \left[\frac{\nu_o \cos\phi}{s}\right] v_o \\
& + C_1 \left[\frac{1}{\rho} + \frac{\nu_o \sin\phi}{s}\right] u_o - \frac{E_o \alpha_o T_o}{1-\nu_o} - \left[\frac{\nu_o}{1-\nu_o}\right] \left[\left(\frac{p_1+p_2}{2}\right) + \left(\frac{h}{2\rho}\right)^2 \sigma_{rr_2}\right]
\end{aligned}
\right\} \text{Zero-order terms}
$$

$$
\left.
\begin{aligned}
& + C_2 \left[\frac{z}{\rho^2}\right] \frac{\partial v_o}{\partial \phi} - C_1 \left[\frac{z}{\rho^2}\right] \left[u_o + \frac{\partial^2 u_o}{\partial \phi^2} - \frac{\rho}{G_o} \frac{\partial \sigma_{r\phi_o}}{\partial \phi}\right] - C_3 \left[\frac{z}{\rho s}\right] \frac{\partial w_o}{\partial \theta} \\
& - C_1 \left[\frac{\nu_o z}{s}\right] \left[\frac{\sin\phi \cos\phi}{s} v_o + \frac{\sin^2\phi \; u_o}{s} + \frac{1}{s} \frac{\partial^2 u_o}{\partial \theta^2} + \frac{1}{G_o} \frac{\partial \sigma_{r\theta_o}}{\partial \theta}\right] \\
& + C_3 \left[\frac{z}{\rho} \frac{\cos\phi}{s}\right] v_o + C_1 \left[\frac{\nu_o z}{\rho} \frac{\cos\phi}{s}\right] \left[v_o - \frac{\partial u_o}{\partial \phi} + \frac{\rho}{G_o} \sigma_{r\phi_o}\right] \\
& + \left[C_2 \left(\frac{z}{\rho^2}\right) + C_3 \left(\frac{z}{\rho} \frac{\sin\phi}{s}\right)\right] u_o \\
& + C_1 \left[\frac{z}{\rho^2} + \frac{\nu_o z}{\rho} \frac{\sin\phi}{s}\right] \left\{ \frac{-\nu_o}{1-\nu_o} \left[\frac{\partial v_o}{\partial \phi} + u_o + \frac{\rho}{s} \left(u_o \sin\phi + v_o \cos\phi - \frac{\partial w_o}{\partial \theta}\right)\right] \right. \\
& \left. + \left[\frac{\rho}{E_o} \frac{(1-2\nu_o)(1+\nu_o)}{1-\nu_o}\right] \sigma_{rr_o} + \left(\frac{1+\nu_o}{1-\nu_o}\right) \rho \alpha_o T_o \right\} \\
& - \left(\frac{z}{\rho}\right) C_4 + \left(\frac{z}{h}\right) \left[\frac{\nu_o}{1-\nu_o}\right] \left[(p_1 - p_2)\right] - \frac{z}{\rho} \left[\frac{\nu_1}{(1-\nu_o)^2}\right] \left[\left(\frac{p_1+p_2}{2}\right) + \left(\frac{h}{2\rho}\right)^2 \sigma_{rr_2}\right]
\end{aligned}
\right\} \text{First-order terms} \qquad \text{(G146)}
$$

$$
\begin{aligned}
\sigma_{\theta\theta} = {} & \left.\begin{aligned}
& C_1\left[\frac{\nu_o}{\rho}\right]\frac{\partial v_o}{\partial\phi} - C_1\left[\frac{1}{s}\right]\frac{\partial w_o}{\partial\theta} + C_1\left[\frac{\cos\phi}{s}\right] v_o \\
& + C_1\left[\frac{\sin\phi}{s}+\frac{\nu_o}{\rho}\right]u_o - \frac{E_o\alpha_o T_o}{1-\nu_o} - \left[\frac{\nu_o}{1-\nu_o}\right]\left[\left(\frac{p_1+p_2}{2}\right)+\left(\frac{h}{2\rho}\right)^2\sigma_{rr_2}\right]
\end{aligned}\right\}\text{Zero-order terms} \\
& \left.\begin{aligned}
& + C_3\left[\frac{z}{\rho^2}\right]\frac{\partial v_o}{\partial\phi} - C_1\left[\frac{\nu_o z}{\rho^2}\right]\left[u_o + \frac{\partial^2 u_o}{\partial\phi^2} - \frac{\rho}{G_o}\frac{\partial\sigma_{r\phi_o}}{\partial\phi}\right] - C_2\left[\frac{z}{\rho s}\right]\frac{\partial w_o}{\partial\theta} \\
& - C_1\left[\frac{z}{s}\right]\left[\frac{\sin^2\phi}{s}u_o + \frac{\rho}{s}\frac{\partial^2 u_o}{\partial\theta^2} + \frac{\rho}{G_o}\frac{\partial\sigma_{r\theta_o}}{\partial\theta}\right] + C_1\left[\frac{z\cos\phi}{\rho s}\right]\cdot \\
& \quad \left[\left(1-\frac{\rho}{s}\right)v_o - \frac{\partial u_o}{\partial\phi} + \frac{\rho}{G_o}\sigma_{r\phi_o}\right] \\
& + C_2\left[\frac{z\cos\phi}{\rho s}\right]v_o + C_1\left[\frac{z\sin\phi}{\rho s}+\frac{\nu_o z}{\rho^2}\right]\Bigg\{\frac{-\nu_o}{1-\nu_o}\Bigg[\frac{\partial v_o}{\partial\phi} + u_o + \frac{\rho}{s}\Bigg(u_o\sin\phi \\
& + v_o\cos\phi - \frac{\partial w_o}{\partial\theta}\Bigg)\Bigg] + \left[\frac{\rho}{E_o}\frac{(1-2\nu_o)(1+\nu_o)}{1-\nu_o}\right]\sigma_{rr_o} + \left(\frac{1+\nu_o}{1-\nu_o}\right)\rho\alpha_o T_o\Bigg\} \\
& + C_2\left[\frac{\sin\phi}{s} + C_3\frac{1}{\rho}\right]\left[\frac{z}{\rho}\right]u_o - \left[\frac{z}{\rho}\right]C_4 + \frac{z}{h}\left[\frac{\nu_o}{1-\nu_o}\right]\left[(p_1-p_2)\right] \\
& - \frac{z}{\rho}\left[\frac{\nu_1}{(1-\nu_o)^2}\right]\left[\left(\frac{p_1+p_2}{2}\right)+\left(\frac{h}{2\rho}\right)^2\sigma_{rr_2}\right]
\end{aligned}\right\}\text{First-order terms}
\end{aligned}
\qquad (G147)
$$

$$
\begin{aligned}
\sigma_{\phi\theta} = {} & \left. - C_5 \left[\frac{1}{s}\right] \frac{\partial v_o}{\partial \theta} + C_5 \left[\frac{1}{\rho}\right] \frac{\partial w_o}{\partial \phi} - C_5 \left[\frac{\cos\phi}{s}\right] w_o \qquad \right\} \text{Zero-order terms} \\
& \left. \begin{aligned}
& - C_6 \left[\frac{z}{\rho s}\right] \frac{\partial v_o}{\partial \theta} - C_5 \left[\frac{z}{\rho s}\right] \left[\left(1 - \frac{\rho}{s} \sin\phi\right) \frac{\partial v_o}{\partial \theta} - \frac{\partial^2 u_o}{\partial \phi \partial \theta} + \frac{\rho}{G_o} \frac{\partial \sigma_{r\phi_o}}{\partial \theta} \right] \\
& + C_6 \left[\frac{z}{\rho^2}\right] \frac{\partial w_o}{\partial \phi} + C_5 \left[\frac{z}{\rho}\right] \Bigg[\left(\frac{s \cos\phi + (\rho - r) \sin\phi \cos\phi}{s^2} \right) w_o \\
& + \left(\frac{\sin\phi}{s} - \frac{1}{\rho} \right) \frac{\partial w_o}{\partial \phi} + \frac{1}{s} \frac{\partial^2 u_o}{\partial \theta \partial \phi} - \frac{r \cos\phi}{s^2} \frac{\partial u_o}{\partial \theta} + \frac{1}{G_o} \frac{\partial \sigma_{r\theta_o}}{\partial \phi} \Bigg] \\
& - C_6 \left[\frac{z}{\rho s} \cos\phi\right] w_o - C_5 \left[\frac{z}{s} \cos\phi\right] \left[\frac{\sin\phi}{s} w_o + \frac{1}{s} \frac{\partial u_o}{\partial \theta} + \frac{1}{G_o} \sigma_{r\theta_o} \right]
\end{aligned} \right\} \text{First-order terms}
\end{aligned}
\tag{G148}
$$

where

$$ s = a + r \sin\phi \tag{G149} $$

$$ C_1 = \frac{E_o}{1 - \nu_o^{\,2}} \tag{G150} $$

$$ C_2 = \frac{E_1}{1 - \nu_o^{\,2}} + \frac{2E_o \nu_o \nu_1}{(1 - \nu_o^{\,2})^2} \tag{G151} $$

$$ C_3 = \frac{E_1 \nu_o}{1 - \nu_o^{\,2}} + \frac{E_o \nu_1 \, (1 + \nu_o^{\,2})}{(1 - \nu_o^{\,2})^2} \tag{G152} $$

$$ C_4 = \frac{E_o \alpha_o T_1}{1 - \nu_o} + \frac{E_o \alpha_1 T_o}{1 - \nu_o} + \frac{E_1 \alpha_o T_o}{1 - \nu_o} + \frac{E_o \alpha_o T_o \nu_1}{(1 - \nu_o)^2} \tag{G153} $$

$$C_5 = \frac{E_o}{1 + \nu_o} \tag{G154}$$

$$C_6 = \frac{E_1}{1 + \nu_o} - \frac{E_o \nu_1}{(1 + \nu_o)^2} \tag{G155}$$

B. THIN SHELL SOLUTION: FINITE DIFFERENCE FORMULATION

The generalized finite-difference analog formulation is directly applicable to the thin-shell solution with the following changes: coordinate α_1 will be eliminated since it corresponds to the radial direction,* and subscript i will be dropped to conform with the above statement.

The finite-difference solution of the equilibrium equations will first be obtained for the general case where the grid spacing is assumed to be irregular. Then a solution will also be obtained for regular grid spacing. In both cases the first, second, and mixed derivatives are needed for the central, forward, and backward grid combinations. A typical general grid spacing is shown in Figure G5.

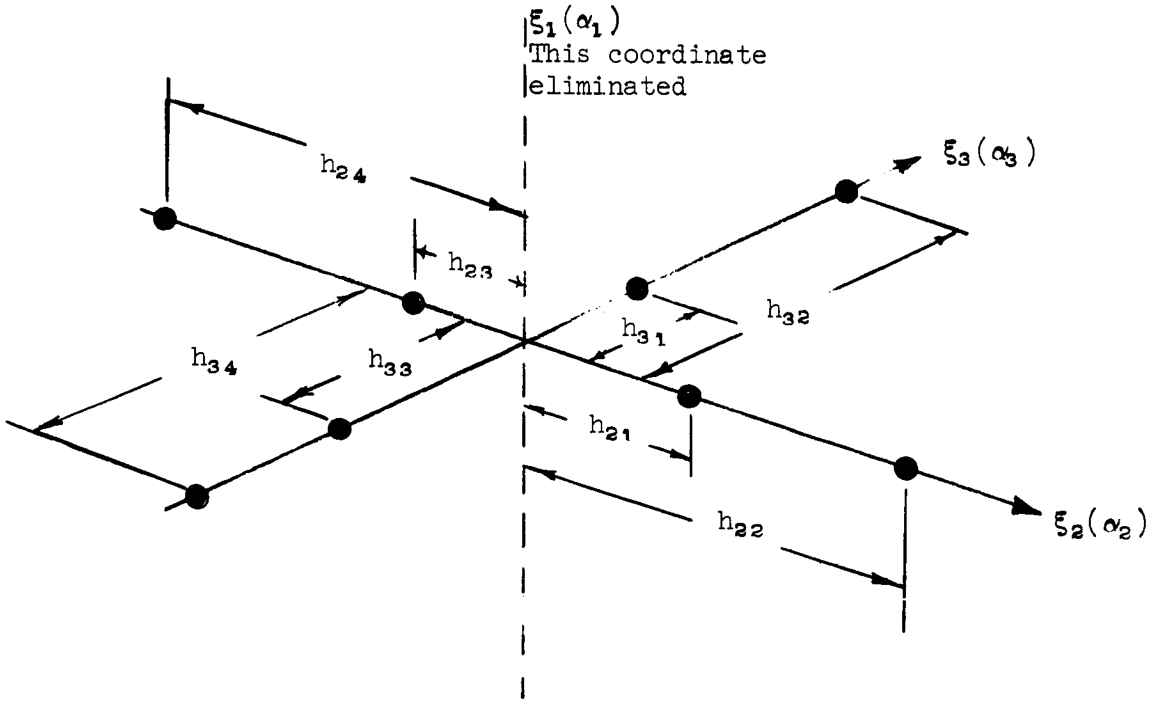


Fig.G5 - Coordinates of Irregular Mesh Intervals

*The partial derivatives in the equilibrium equations given in Equation (G36) are taken with respect to α_2 and α_3 (ϕ and Θ, respectively).

The increments in the vicinity of a node will be designated by the following notations, in accordance with Figure G5.

$$
\begin{aligned}
h_{21} &= (\alpha_2)_{i+1} - (\alpha_2)_i & h_{31} &= (\alpha_3)_{i+1} - (\alpha_3)_i \\
h_{22} &= (\alpha_2)_{i+2} - (\alpha_2)_i & h_{32} &= (\alpha_3)_{i+2} - (\alpha_3)_i \\
h_{23} &= (\alpha_2)_i - (\alpha_2)_{i-1} & h_{33} &= (\alpha_3)_i - (\alpha_3)_{i-1} \\
h_{24} &= (\alpha_2)_i - (\alpha_2)_{i-2} & h_{34} &= (\alpha_3)_i - (\alpha_3)_{i-2}
\end{aligned}
\tag{G156}
$$

From Equation (G36) it can be seen that $f(\xi_1, \xi_2, \xi_3)$, the function of the coordinates with the origin at i, j, k, is

$$
\begin{aligned}
f(\xi_1, \xi_2, \xi_3) = f_{i,j,k} &+ B_1\xi_1 + B_2\xi_2 + B_3\xi_3 + B_4\xi_1\xi_2 + B_5\xi_2\xi_3 + B_6\xi_3\xi_1 \\
&+ B_7\xi_1{}^2 + B_8\xi_2{}^2 + B_9\xi_3{}^2 + B_{10}\xi_1\xi_2\xi_3 + B_{11}\xi_1\xi_2{}^2 \\
&+ B_{12}\xi_1\xi_3{}^2 + B_{13}\xi_1{}^2\xi_2 + B_{14}\xi_2\xi_3{}^2 + \ldots
\end{aligned}
$$

The first and second derivatives of $f(\alpha_1, \alpha_2, \alpha_3)$ with respect to α_2 and α_3 are obtained from Equation (G36).

$$
\left.\frac{\partial f}{\partial \alpha_2}\right|_{i,j,k} = \left.\frac{\partial f}{\partial \xi_2}\right|_{o,o,o} = B_2 \ , \quad \left.\frac{\partial f}{\partial \alpha_3}\right|_{i,j,k} = \left.\frac{\partial f}{\partial \xi_3}\right|_{o,o,o} = B_3 \ ,
$$

$$
\left.\frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_3}\right|_{i,j,k} = \left.\frac{\partial^2 f}{\partial \xi_2 \partial \xi_3}\right|_{o,o,o} = B_5 \ , \quad \left.\frac{\partial^2 f}{\partial \alpha_2{}^2}\right|_{i,j,k} = \left.\frac{\partial^2 f}{\partial \xi_2{}^2}\right|_{o,o,o} = 2B_8 \ ,
$$

$$
\left.\frac{\partial^2 f}{\partial \alpha_3{}^2}\right|_{i,j,k} = \left.\frac{\partial^2 f}{\partial \xi_3{}^2}\right|_{o,o,o} = 2B_9 \tag{G157}
$$

The constants B_i are evaluated in terms of the function at these nodes and the grid spacing as shown in Figure G5 by considering the values of $f(\xi_1, \xi_2, \xi_3)$ at the eight nodes adjacent to j,k, and proceeding in a manner similar to that outlined on pages G30 and G31.

1. General Case - Irregular Grid Spacing

a. Central Derivatives

$$\left.\frac{\partial f}{\partial \alpha_2}\right|_{j,k} = \frac{h_{23}{}^2 f_{j+1,k} + (h_{21}{}^2 - h_{23}{}^2)\, f_{j,k} - h_{21}{}^2 f_{j-1,k}}{h_{21} h_{23}\, (h_{21} + h_{23})} \tag{G158}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right|_{j,k} = \frac{h_{33}{}^2 f_{j,k+1} + (h_{31}{}^2 - h_{33}{}^2)\, f_{j,k} - h_{31}{}^2 f_{j,k-1}}{h_{31} h_{33}\, (h_{31} + h_{33})} \tag{G159}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_3}\right|_{j,k} = \frac{1}{h_{31} h_{33}\, (h_{21} + h_{23})(h_{31} + h_{33})} \Big[h_{33}{}^2\, (f_{j+1,k+1} - f_{j-1,k+1}) - (h_{33}{}^2 - h_{31}{}^2)(f_{j+1,k} - f_{j-1,k}) - h_{31}{}^2\, (f_{j+1,k-1} - f_{j-1,k-1}) \Big] \tag{G160}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_3}\right|_{j,k} = \frac{1}{h_{21} h_{23}\, (h_{21} + h_{23})(h_{31} + h_{33})} \Big[h_{23}{}^2\, (f_{j+1,k+1} - f_{j+1,k-1}) - (h_{23}{}^2 - h_{21}{}^2)(f_{j,k+1} - f_{j,k-1}) - h_{21}{}^2\, (f_{j-1,k+1} - f_{j-1,k-1}) \Big] \tag{G161}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2{}^2}\right|_{j,k} = \frac{2\left[h_{23} f_{j+1,k} - (h_{21} + h_{23})\, f_{j,k} + h_{21} f_{j-1,k} \right]}{h_{21} h_{23}\, (h_{21} + h_{23})} \tag{G162}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3{}^2}\right|_{j,k} = \frac{2\left[h_{33} f_{j,k+1} - (h_{31} + h_{33})\, f_{j,k} + h_{31} f_{j,k-1} \right]}{h_{31} h_{33}\, (h_{31} + h_{33})} \tag{G163}$$

b. Forward Derivatives

$$\left.\frac{\partial f}{\partial \alpha_2}\right|_{j,k} = \frac{h_{22}{}^2 f_{j+1,k} - (h_{22}{}^2 - h_{21}{}^2)\, f_{j,k} - h_{21}{}^2 f_{j+2,k}}{h_{21} h_{22}\, (h_{22} - h_{21})} \tag{G164}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right|_{j,k} = \frac{h_{32}^{2} f_{j,k+1} - (h_{32}^{2} - h_{31}^{2})\, f_{j,k} - h_{31}^{2} f_{j,k+2}}{h_{31} h_{32}\, (h_{32} - h_{31})} \tag{G165}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_3}\right|_{j,k} = \frac{1}{h_{31} h_{33}\, (h_{21} - h_{22})(h_{31} + h_{33})} \Big[h_{33}^{2}\, (f_{j+1,k+1} - f_{j+1,k} - f_{j+2,k+1} + f_{j+2,k}) - h_{31}^{2}\, (f_{j+1,k-1} - f_{j+1,k} - f_{j+2,k-1} + f_{j+2,k}) \Big] \tag{G166}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_3}\right|_{j,k} = \frac{1}{h_{21} h_{23}\, (h_{31} - h_{32})(h_{21} + h_{23})} \Big[h_{23}^{2}\, (f_{j+1,k+1} - f_{j,k+1} - f_{j+1,k+2} + f_{j,k+2}) - h_{21}^{2}\, (f_{j-1,k+1} - f_{j,k+1} - f_{j-1,k+2} + f_{j,k+2}) \Big] \tag{G167}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2^{2}}\right|_{j,k} = \frac{2 \left[- h_{22} f_{j+1,k} + (h_{22} - h_{21})\, f_{j,k} + h_{21} f_{j+2,k} \right]}{h_{21} h_{22}\, (h_{22} - h_{21})} \tag{G168}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^{2}}\right|_{j,k} = \frac{2 \left[- h_{32} f_{j,k+1} + (h_{32} - h_{31})\, f_{j,k} + h_{31} f_{j,k+2} \right]}{h_{31} h_{32}\, (h_{32} - h_{31})} \tag{G169}$$

c. Backward Derivatives

$$\left.\frac{\partial f}{\partial \alpha_2}\right|_{j,k} = \frac{h_{23}^{2} f_{j-2,k} + (h_{24}^{2} - h_{23}^{2})\, f_{j,k} - h_{24}^{2} f_{j-1,k}}{h_{23} h_{24}\, (h_{24} - h_{23})} \tag{G170}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right|_{j,k} = \frac{h_{33}^{2} f_{j,k-2} + (h_{34}^{2} - h_{33}^{2})\, f_{j,k} - h_{34}^{2} f_{j,k-1}}{h_{33} h_{34}\, (h_{34} - h_{33})} \tag{G171}$$

$$\left.\frac{\partial^2 f}{\partial\alpha_2\partial\alpha_3}\right|_{j,k} = \frac{1}{h_{31}h_{33}\,(h_{23} - h_{24})(h_{31} + h_{33})}\Big[h_{33}^{\ 2}\,(f_{j-1,k+1} - f_{j-2,k+1}$$

$$+ f_{j-2,k} - f_{j-1,k}) - h_{31}^{\ 2}\,(f_{j-1,k-1} - f_{j-1,k} - f_{j-2,k-1} + f_{j-2,k})\Big] \tag{G172}$$

$$\left.\frac{\partial^2 f}{\partial\alpha_2\partial\alpha_3}\right|_{j,k} = \frac{1}{h_{21}h_{23}\,(h_{33} - h_{34})(h_{21} + h_{23})}\Big[h_{23}^{\ 2}\,(f_{j+1,k-1} - f_{j+1,k-2}$$

$$+ f_{j,k-2} - f_{j,k-1}) - h_{21}^{\ 2}\,(f_{j-1,k-1} - f_{j,k-1} - f_{j-1,k-2} + f_{j,k-2})\Big] \tag{G173}$$

$$\left.\frac{\partial^2 f}{\partial\alpha_2^{\ 2}}\right|_{j,k} = \frac{2\left[h_{23}f_{j-2,k} + (h_{24} - h_{23})\,f_{j,k} - h_{24}f_{j-1,k}\right]}{h_{23}h_{24}\,(h_{24} - h_{23})} \tag{G174}$$

$$\left.\frac{\partial^2 f}{\partial\alpha_3^{\ 2}}\right|_{j,k} = \frac{2\left[h_{33}f_{j,k-2} + (h_{34} - h_{33})\,f_{j,k} - h_{34}f_{j,k-1}\right]}{h_{33}h_{34}\,(h_{34} - h_{33})} \tag{G175}$$

2. General Case - Regular Grid Spacing

When the grid spacing is regular, then

$$\begin{aligned} h_2 &= h_{21} = h_{23} = \tfrac{1}{2}\,h_{22} = \tfrac{1}{2}\,h_{24} \\ h_3 &= h_{31} = h_{33} = \tfrac{1}{2}\,h_{32} = \tfrac{1}{2}\,h_{34} \end{aligned} \tag{G176}$$

Substituting the conditions of Equation (G176) into Equations (G158) through (G175) the first, second, and mixed derivatives for the central, forward, and backward regular grid spacing combinations are obtained.

a. Central Derivatives

$$\left.\frac{\partial f}{\partial\alpha_2}\right|_{j,k} = \frac{f_{j+1,k} - f_{j-1,k}}{2h_2} \tag{G177}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right|_{j,k} = \frac{f_{j,k+1} - f_{j,k-1}}{2h_3} \tag{G178}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_3}\right|_{j,k} = \frac{1}{4h_2h_3}\left[f_{j+1,k+1} - f_{j+1,k-1} - f_{j-1,k+1} + f_{j-1,k-1}\right] \tag{G179}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2^2}\right|_{j,k} = \frac{1}{h_2^2}\left[f_{j+1,k} - 2f_{j,k} + f_{j-1,k}\right] \tag{G180}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^2}\right|_{j,k} = \frac{1}{h_3^2}\left[f_{j,k+1} - 2f_{j,k} + f_{j,k-1}\right] \tag{G181}$$

b. Forward Derivatives

$$\left.\frac{\partial f}{\partial \alpha_2}\right|_{j,k} = \frac{1}{2h_2}\left[4f_{j+1,k} - 3f_{j,k} - f_{j+2,k}\right] \tag{G182}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right|_{j,k} = \frac{1}{2h_3}\left[4f_{j,k+1} - 3f_{j,k} - f_{j,k+2}\right] \tag{G183}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_3}\right|_{j,k} = \frac{-1}{2h_2h_3}\left[f_{j+1,k+1} - f_{j+2,k+1} - f_{j+1,k-1} + f_{j+2,k-1}\right] \tag{G184}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_3}\right|_{j,k} = \frac{-1}{2h_2h_3}\left[f_{j+1,k+1} - f_{j+1,k+2} - f_{j-1,k+1} + f_{j-1,k+2}\right] \tag{G185}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2^2}\right|_{j,k} = \frac{1}{h_2^2}\left[- 2f_{j+1,k} + f_{j,k} + f_{j+2,k}\right] \tag{G186}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^{\,2}}\right|_{j,k} = \frac{1}{h_3^{\,2}}\left[-2f_{j,k+1} + f_{j,k} + f_{j,k+2}\right] \tag{G187}$$

C. Backward Derivatives

$$\left.\frac{\partial f}{\partial \alpha_2}\right|_{j,k} = \frac{1}{2h_2}\left[f_{j-2,k} + 3f_{j,k} - 4f_{j-1,k}\right] \tag{G188}$$

$$\left.\frac{\partial f}{\partial \alpha_3}\right|_{j,k} = \frac{1}{2h_3}\left[f_{j,k-2} + 3f_{j,k} - 4f_{j,k-1}\right] \tag{G189}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_3}\right|_{j,k} = \frac{-1}{2h_2h_3}\left[f_{j-1,k+1} - f_{j-2,k+1} - f_{j-1,k-1} + f_{j-2,k-1}\right] \tag{G190}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_2 \partial \alpha_3}\right|_{j,k} = \frac{-1}{2h_2h_3}\left[f_{j+1,k-1} - f_{j+1,k-2} - f_{j-1,k-1} + f_{j-1,k-2}\right] \tag{G191}$$

$$\left.\frac{\partial^2 f}{\partial \alpha_3^{\,2}}\right|_{j,k} = \frac{1}{h_3^{\,2}}\left[f_{j,k-2} + f_{j,k} - 2f_{j,k-1}\right] \tag{G193}$$

V. OVERALL THREE-DIMENSIONAL BOUNDARY CONDITIONS

Given the notation and equations previously developed, and with Figures G6a, G6b, and G6c illustrating the geometry involved, the following described boundary and compatibility conditions are those which will govern the solution to the full three-dimensional heat shield problem for the two significant cases ("fixed" and "free-free" conditions, respectively, at the structural juncture surface).

A. On inside and outside boundary surfaces OB and IB for both cases:

$$\tau_{rr} = \tau_{r\phi} = \tau_{r\theta} = 0$$

B. On interface surfaces I1 and I2 for both cases (index denotes which material medium is indicated):

$$\left.\begin{aligned} \left[\tau_{rr}\right]_{i+1} &= \left[\tau_{rr}\right]_{i} \\ \left[\tau_{r\phi}\right]_{i+1} &= \left[\tau_{r\phi}\right]_{i} \\ \left[\tau_{r\theta}\right]_{i+1} &= \left[\tau_{r\theta}\right]_{i} \end{aligned}\right\} \quad i = 1,2$$

C. In the $r - \phi$ plane for which $\theta = 0^{\circ}$ for both cases:

$$w = \frac{\partial^2 w}{\partial \theta^2} = \frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial \theta} = 0$$

D. In the $r - \phi$ plane for which $\theta = 90^{\circ}$ for both cases:

$$u(r,\phi,90^{\circ}) = u(r,-\phi,90^{\circ})$$

$$v(r,\phi,90^{\circ}) = v(r,-\phi,90^{\circ})$$

$$w(r,\phi,90^{\circ}) = -w(r,-\phi,90^{\circ})$$

$$\left[\frac{\partial u}{\partial \theta}\right]_{\phi=\phi_1} = \left[-\frac{\partial u}{\partial \theta}\right]_{\phi=-\phi_1}$$

$$\left[\frac{\partial v}{\partial \theta}\right]_{\phi=\phi_1} = \left[-\frac{\partial v}{\partial \theta}\right]_{\phi=-\phi_1}$$

$$\left[\frac{\partial w}{\partial \theta}\right]_{\phi=\phi_1} = \left[-\frac{\partial w}{\partial \theta}\right]_{\phi=-\phi_1}$$

E. On the structural juncture plane (SJ) for the "fixed" case:

$$u = v = w = 0$$

F. On the structural juncture plane (SJ) for the "free-free" case:

$$\tau_{\phi\phi} = \tau_{r\phi} = \tau_{\theta\phi} = 0$$

G. Along the circles at the intersections between the structural juncture plane and each interface surface for the "free-free" case (IP1, IP2), the conditions of Condition F are replaced by (index denotes which material medium is indicated):

$$\left[\frac{\partial v}{\partial r}\right]_{i+1} = \left[\frac{\partial v}{\partial r}\right]_{i}$$

$$\left[\frac{\partial w}{\partial r}\right]_{i+1} = \left[\frac{\partial w}{\partial r}\right]_{i}$$

$$\left[\frac{\partial u}{\partial r}\right]_{i+1} - \left[\frac{\partial u}{\partial r}\right]_{i} + \left(\frac{2}{r}\right)\left\{\left[\frac{\mu}{\lambda}\right]_{i+1} - \left[\frac{\mu}{\lambda}\right]_{i}\right\}\left(u + \frac{\partial v}{\partial \phi}\right) \qquad i = 1,2$$

$$+ \left[\left(3 + \frac{2\mu}{\lambda}\right)\int_{T_o}^{T} \alpha dT\right]_{i+1} - \left[\left(3 + \frac{2\mu}{\lambda}\right)\int_{T_o}^{T} \alpha dT\right]_{i} = 0$$

H. Along the circles at the intersections between the structural juncture plane and each of the boundary surfaces OB and IB for the "free-free" case (PI,PO), the normal stress condition ($\tau_{\phi\phi} = 0$) of Condition F is replaced by

$$u = r\frac{\partial u}{\partial r} - \frac{\partial v}{\partial \phi}$$

AB is edge view of plane
of temperature and thickness
distributional symmetry.

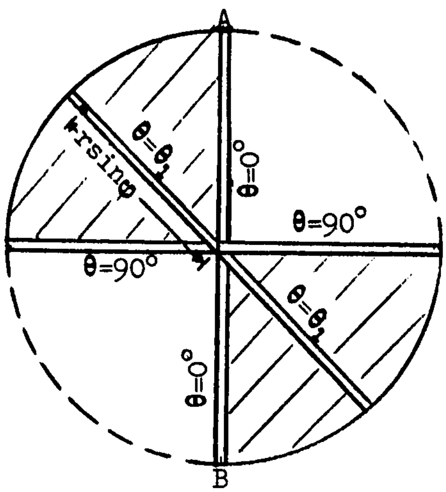


Fig. G6a - Front View of Shield

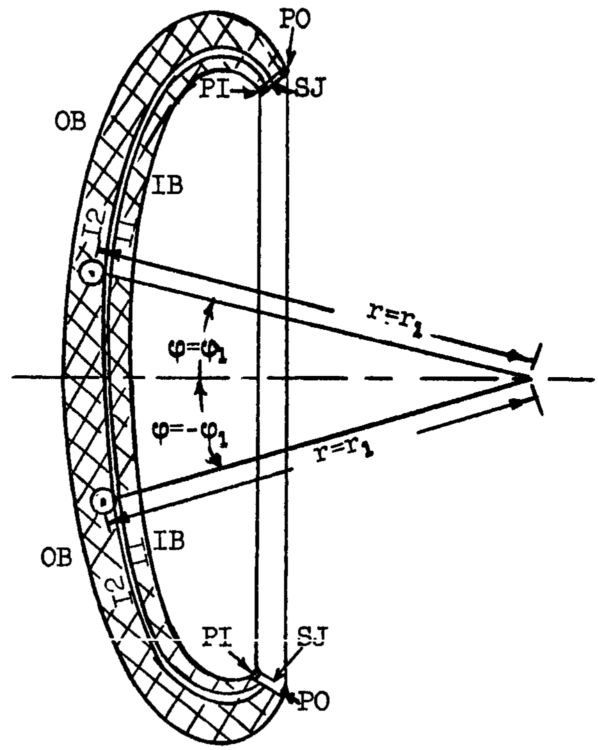


Fig. G6b - Cross Section of Shield

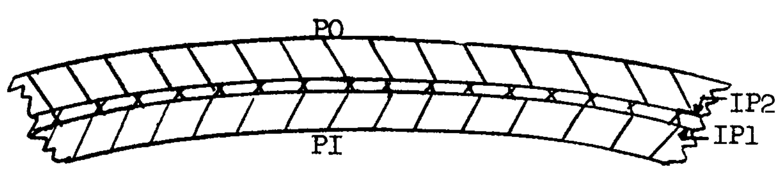


Fig. G6c - Portion of Structural Juncture Surface SJ

Definitions:

r - Radial direction (both sphere and torus)
IB - Inside boundary surface
OB - Outside boundary surface
SJ - Structural juncture plane
IP1, IP2 - Intersections between the structural juncture plane and each surface interface
PI, PO - Intersections between the structural juncture plane and each of the boundary surfaces OB and IB

VI. PROGRAMING OF THE FINITE-DIFFERENCE MODEL(S) AND ATTEMPTS TO SOLVE THE EQUATIONS BY THE OVER-RELAXATION APPROACH AND BY DIRECT MATRIX INVERSION

A. OVER-RELAXATION APPROACH

The first approach in an attempt to solve the displacement and stress equations was point relaxation. The reason for this choice was the apparent success of this method in the previous work by Morgan and Christensen. Repeated attempts with different values of the over-relaxation experiments were not successful. Evidently the reason for the difficulty in point relaxation was the incapability of this method to bring in the effects of boundary conditions.

An alternative form (i.e., line relaxation) was then attempted. The first trial with line relaxation was made utilizing a radial line to the boundary. Repeated attempts were made with this method with various boundary conditions. It was demonstrated that this method is also incapable of meeting the remaining boundary conditions. A "long line" was then attempted. It was again demonstrated that all boundary conditions could not be met simultaneously; the solution diverged as it proceeded radially outward.

The basic difficulty in these methods seems to lie in the formulation of an acceptable and consistent system of boundary conditions. Similar difficulties have been reported in the literature for very much simpler cases.

B. DIRECT MATRIX INVERSION

A second approach (i.e., direct matrix inversion) was then attempted. Due to the relatively small size of the computer memory, the mesh size was too large to achieve a successful solution. However, direct matrix inversion cannot be ruled out as a method for the solution of this problem. It should be noted that the very short time allotted to the attempts at the solution using relaxation and direct inversion methods did not allow a complete exploration of these procedures.

VII. ALTERNATIVE METHODS AND RECOMMENDATIONS

Additional methods for the solution of the complete nonaxisymmetric case were investigated. It was determined that the possibility exists for a direct matrix inversion solution of the whole problem provided machine language is used throughout for the programing. An additional memory capability utilizing magnetic tape or memory disks would be employed.

A second approach was also investigated which would utilize equivalent analog circuits to transform the equations into a set, the behavior of which is well known. This approach seems to offer another possibility of solving the complete three-dimensional nonaxisymmetric problem.